Universidade de Aveiro Departamento de Matemática
2007

José Carlos Alves Martins Aleixo

Sistemas Periódicos Comportamentais

Behavioral Periodic Systems

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática. Trabalho realizado sob a orientação científica da Doutora Maria Paula Macedo Rocha Malonek, Professora Catedrática do Departamento de Matemática da Universidade de Aveiro e da Doutora Maria Teresa dos Reis Pedroso de Lima Oliveira, Professora Catedrática da Faculdade de Economia da Universidade de Coimbra.

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Às flores do meu jardim / To the flowers of my garden,

Beatriz, Mateus e Francisca

o júri / the jury

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palavras-chave Sistemas Comportamentais, Sistemas Periódicos.

resumo

Esta tese dedica-se ao estudo de sistemas periódicos comportamentais, tendo como base, por um lado, a abordagem clássica aos sistemas periódicos de espaço de estados e, por outro, a abordagem comportamental aos sistemas dinâmicos.

Usando uma formulação invariante no tempo anteriormente proposta na literatura, estabelecem-se vários resultados sobre as propriedades de várias descrições matemáticas (representações) dos comportamentos periódicos. Estudam-se também algumas importantes propriedades destes comportamentos, como a controlabilidade e a observabilidade.

keywords

Behavioral Systems, Periodic Systems.

abstract

This thesis is devoted to the study of behavioral periodic systems, based on the classical approach to periodic state space systems on the one hand, and on the behavioral approach to dynamical systems on the other hand.

Using a time-invariant formulation, which has already been proposed in the literature, some results are obtained as regards to several mathematical descriptions (or representations) of periodic behaviors. Some important properties are also studied such as controllability and observability.

Contents

"If I have ever made any valuable discoveries, it has been owing more to patient attention, than to any other talent."

— Isaac Newton

"He who never made a mistake never made a discovery."

— Samuel Smiles

Introduction

Motivation

This thesis deals with periodic systems in the context of the behavioral approach to dynamical systems. The motivation for this work has its roots in a work published by Margreet Kuijper and Jan C. Willems, see [47], where, citing the authors,

> "*We set out to develop a framework for the analysis and synthesis of discrete-time periodically time-varying systems. Adopting a behavioral approach, we define the concept of periodicity in terms of the trajectories of the system. We subsequently use this framework to investigate several basic notions, such as controllability, on the level of trajectories and also present several techniques for associating time-invariant systems in a behavioral way.*"

This work combines two important elements. First in the list: *discrete-time periodically time-varying systems*. Periodic systems lie, as an intermediate class of systems, somewhere between the time-varying realm and the time-invariant case, which, in turn, can be seen as a particular case of periodicity. The increasing interest in periodic systems is motivated by a huge number of processes which can be modeled by linear models with periodically time-varying coefficients, such as, satellite attitude control based on the periodicity of the earth magnetic field, control of rotating machinery, or sampled-data systems. An overview of the vast literature in the field of linear periodic systems, in the classical input-state-output framework, can be found in, e.g., [9, 11–30, 32–35, 37, 40–44, 48–50, 55, 57, 58, 63] and the references therein. Second in the list: the *behavioral approach*. The behavioral theory of linear, time-invariant systems has its roots in the mid eighties when Jan C. Willems started his pioneering work. The main paradigms are the emphasis on the behavior, the set of possible trajectories of the system, rather than on its mathematical representations, and the absence of an *a prior* partition of the signals into inputs and outputs. Nowadays, the behavioral approach has reached a fair level of maturity with branches in partial differential equations, discrete event systems, computer science and coding theory. The reason for this diversity of applications is that all concepts are defined and studied in terms of the behavior and on a level of abstraction that indeed enables translations in many different contexts. An example in this respect is the notion of controllability. The classical definition is stated for input-output systems in state space form. The behavioral definition of controllability, however, is more abstract and therefore more general. It can be applied directly to systems of various natures, see [38] for a convincing account.

Returning back to the classical discrete-time periodically time-varying systems, most of the usual approaches have as a common factor the use of *time-invariant descriptions*. In fact, the correspondence between periodic and time-invariant systems represents a powerful tool both for analysis and for control purposes, since it allows us to restate several problems in a time-invariant context. Citing [27],

> "*Most analysis and control problems for discrete-time periodic systems can be equivalently recast as problems in the realm of time-invariant systems by exploiting the existing isomorphism between the two classes of systems. The most popular, among all the possible equivalent time-invariant representations of a periodic system is the so-called* lifted reformulation."

Different techniques have been proposed in the literature to obtain time-invariant formulations for periodic systems. Tracing back in time, the first appearance of such techniques may be linked to a work of Kranc, see [44], and related works as [30]. But it was only with the works of Meyer and Burrus, see [48, 49], and Khargonekar, Poolla and Tannenbaum, see [42], that a "primitive" version of the so-called lifted reformulation made its first steps, although without this designation. The origin for this term, *lifting*, is due to Yamamoto, see [62]. Roughly speaking, the lifting of a periodic system is a time-invariant system whose input-output trajectories are obtained by stacking the trajectories of the original system together with an adequate number of its successive shifts. An alternative to this *lifting technique* may be found in Ana Urbano's PhD thesis, [55]. Here, a periodic state space system is associated not with one, but with a suitable number of time-invariant state space systems.

More recently another equivalent time-invariant reformulation, commonly referred to as *cyclic reformulation*, was introduced in the works of Verriest, see [58], Park and

Verriest, see [50], and Flamm, see [31]. This differs from the lifting approach, as the associated invariant system is obtained as consisting of a suitable number of shuffled copies of the original periodic system, operating independently in parallel.

A condensed overview of the lifted and cyclic reformulations may be found in the paper of Colaneri and Kučera, [27].

Based on these tools, Margreet Kuijper and Jan C. Willems, have developed techniques in order to connect behavioral periodic systems with time-invariant lifted and cyclic (renamed, twisted) formulations, see [46].

As a starting point for this thesis we recall the conclusions presented by Margreet Kuijper and Jan C. Willems in [47]:

> "*In this paper we have introduced and investigated several system theoretic notions for periodically time-varying systems on the level of the system's trajectories. We have also addressed the question: how do these notions express themselves in terms of a representation of the system? Here the type of representation used is more general than usually considered in the periodically time-varying literature. The type studied is the natural one that comes up in a behavioral framework. It is a topic of future research to investigate this type of representation in more detail as well as exploit the presented "lifting" and "twisting" techniques further.*"

The aim of our work is precisely to deepen the study of representations for periodic behaviors, as well as the study of some important system structural properties that may be reflected into the features of such representations.

Outline of the thesis

We now give a brief summary of the contents of each chapter of this thesis.

Chapter 1 – Periodic state space systems

We review some basic concepts concerning periodic state space systems. More concretely, based on [37,55], we present a time-invariant dynamical decomposition (formulation) for these systems and recall structural properties as controllability, reachability, reconstructibility and observability.

Chapter 2 – Time-invariant behaviors

This chapter contains an overview of the background material concerning time-invariant behaviors in order to more easily establish the connection between periodic systems and their associated time-invariant formulations, to be introduced later on. Some of these subjects are well-known within the behavioral approach while others have been developed during our research.

Chapter 3 – Periodic behaviors and their representations

We focus on periodic behaviors, which allow a kernel-type representation, called *P-periodic kernel representation (P-PKR)*. An important tool in the study carried out here is the *lifted behavior* introduced in [47], which is a time-invariant behavior whose trajectories are constructed from the trajectories of the original periodic behavior, similar to what happens in [55] within the classical approach. Based on the relation between the representations of periodic behaviors and the representations of the associated time-invariant behaviors obtained by lifting, we characterize P-periodic kernel representations with respect to equivalence and minimality. Further, we introduce latent variable (and, in particular, image) representations in the periodic context and obtain a latent variable elimination procedure using lifted behaviors.

Chapter 4 – Controllability, autonomy and free variables

Using the definition of behavioral controllability, it is possible to obtain a correspondence between the controllability of a periodic system and of its associated lifted system. This is the key tool that, together with known results for the time-invariant case, enables us to characterize the controllability of periodic systems. The obtained results are applied to the particular case of periodic state space systems, namely in what concerns the relation between state space and behavioral controllability, leading to similar conclusions as for the time-invariant case. An autonomy characterization for periodic behaviors is obtained based on the connection established between the autonomy of a periodic system and the associated lifted system. We also prove the existence of an autonomous/controllable decomposition similar to what happens in the time-invariant case. Finally, we introduce a new concept of free variables and inputs, which can be regarded as a generalization of the one adopted for time-invariant systems, but appears to be more adequate for the periodic case.

Chapter 5 – Reconstructibility and observability

Using the definition of behavioral reconstructibility stated in Chapter 2, we obtain a correspondence between the reconstructibility of a periodic system and the reconstructibility of its associated lifted system. This is the key tool that enables the characterization of reconstructibility of periodic systems, by using known results for the time-invariant case. These results are applied to the particular case of periodic state space systems, in order to analyse the relationship between the behavioral and the classical reconstructibility notions, leading to similar characterizations as for the case of time-invariant state space systems. Further, we prove the equivalence between the notions of *Willems-observability* and reconstructibility for periodic behaviors, as happens for time-invariant behaviors.

The last chapter, of conclusions, is devoted to summarizing the main results that are contained in this thesis. This is made in a very brief way, since every other chapter already contains its own conclusions.

Part A

Background

"An invasion of armies can be resisted, but not an idea whose time has come."

— Victor Hugo

"Man's mind, once stretched by a new idea, never regains its original dimensions."

— Oliver Wendell Holmes

1

Periodic state space systems

We review some basic concepts concerning periodic state space systems. More concretely, based on [37, 55], we present a time-invariant dynamical decomposition (formulation) for these systems and recall structural properties as controllability, reachability, reconstructibility and observability.

§1.1 Introduction

The classical state space approach to P-periodic systems takes as its starting point a description of the form:

$$\begin{cases} x(k+1) = A(k)\,x(k) + B(k)\,u(k) \\ \quad\; y(k) = C(k)\,x(k) + D(k)\,u(k) \end{cases} \qquad k \in \mathbb{Z}, \tag{1.1}$$

where the matrices $A(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times m}$, $C(k) \in \mathbb{R}^{p\times n}$ and $D(k) \in \mathbb{R}^{p\times m}$ are periodic functions of k with period $P \in \mathbb{N}$, x is the state variable and u and y are the input and output, respectively. From here on we will refer to this system in short as Σ_s or alternatively as $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$. Time invariance can be regarded as a particular case of periodicity, with period $P = 1$. In this case $A(k) \equiv A$, $B(k) \equiv B$, $C(k) \equiv C$, $D(k) \equiv D$, for $k \in \mathbb{Z}$, and we use the notation (A, B, C, D).

§1.2 Invariant dynamical decomposition

In [55] and [37] an invariant dynamical decomposition associated with the P-periodic state system description (1.1) is introduced allowing a one-to-one correspondence between a P-periodic state space system and P time-invariant ones.

In order to get some motivation and insight of this method we will first consider the example given by a 2-periodic system. Let us split the time axis into even and odd instants, and analyse the system dynamics in these separate instants.

Note that then, from the system dynamics equation and due to the 2-periodicity of the matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$, we obtain two cases, namely:

i) *even case*

$$\begin{aligned}
x(2) &= A(1)\,x(1) + B(1)\,u(1) \\
&= A(1)\,(A(0)\,x(0) + B(0)\,u(0)) + B(1)\,u(1) \\
&= A(1)\,A(0)\,x(0) + \begin{bmatrix} B(1) & A(1)\,B(0) \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \\
y(2) &= C(2)\,x(2) + D(2)\,u(2) \\
&= C(0)\,(A(1)\,x(1) + B(1)\,u(1)) + D(0)\,u(2) \\
&= C(0)\,A(1)\,x(1) + \begin{bmatrix} D(0) & C(0)\,B(1) \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \end{bmatrix} \\
&\vdots \\
x(2(k+1)) &= A(1)\,A(0)\,x(2k) + \begin{bmatrix} B(1) & A(1)\,B(0) \end{bmatrix} \begin{bmatrix} u(2k+1) \\ u(2k) \end{bmatrix} \\
y(2k) &= C(0)\,A(1)\,x(2k-1) + \begin{bmatrix} D(0) & C(0)\,B(1) \end{bmatrix} \begin{bmatrix} u(2k) \\ u(2k-1) \end{bmatrix}
\end{aligned}$$

ii) *odd case*

$$\begin{aligned}
x(3) &= A(0)\,x(2) + B(0)\,u(2) \\
&= A(0)\left(A(1)\,x(1) + B(1)\,u(1)\right) + B(0)\,u(2) \\
&= A(0)\,A(1)\,x(1) + \begin{bmatrix} B(0) & A(0)\,B(1) \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \end{bmatrix} \\
y(3) &= C(3)\,x(3) + D(3)\,u(3) \\
&= C(1)\left(A(0)\,x(2) + B(0)\,u(2)\right) + D(1)\,u(3) \\
&= C(1)\,A(0)\,x(2) + \begin{bmatrix} D(1) & C(1)\,B(0) \end{bmatrix} \begin{bmatrix} u(3) \\ u(2) \end{bmatrix} \\
&\;\vdots \\
x(2k+1) &= A(0)\,A(1)\,x(2(k-1)+1) \\
&\qquad + \begin{bmatrix} B(0) & A(0)\,B(1) \end{bmatrix} \begin{bmatrix} u(2k) \\ u(2(k-1)+1) \end{bmatrix} \\
y(2(k-1)+1) &= C(1)\,A(0)\,x(2(k-1)) \\
&\qquad + \begin{bmatrix} D(1) & C(1)\,B(0) \end{bmatrix} \begin{bmatrix} u(2(k-1)+1) \\ u(2(k-1)) \end{bmatrix}.
\end{aligned}$$

In this way, we obtain two time-invariant systems, namely

$$\Sigma_0 = \left(A(1)\,A(0), [B(1) \quad A(1)\,B(0)], \begin{bmatrix} C(0) \\ C(1)\,A(0) \end{bmatrix}, \begin{bmatrix} 0 & D(0) \\ D(1) & C(1)\,B(0) \end{bmatrix}\right)$$

and

$$\Sigma_1 = \left(A(0)\,A(1), [B(0) \quad A(0)\,B(1)], \begin{bmatrix} C(1) \\ C(0)\,A(1) \end{bmatrix}, \begin{bmatrix} 0 & D(1) \\ D(0) & C(0)\,B(1) \end{bmatrix}\right),$$

that together describe the evolution of the original 2-periodic system. Generalizing from this procedure, given a P-periodic state space system Σ_s described in (1.1), we can define P associated time-invariant systems $\Sigma_t,\ t = 0, \ldots, P-1$, as

$$\Sigma_t \equiv \begin{cases} x_t(k+1) = A_t x_t(k) + B_t u_t(k) \\ \quad y_t(k) = C_t x_t(k) + D_t u_t(k) \end{cases} \qquad k \in \mathbb{Z},$$

where

$$A_t := \phi_A(t+P,t) \tag{1.2}$$

$$B_t := \Big[B(t{+}P{-}1) \quad \phi_A(t{+}P,t{+}P{-}1)\,B(t{+}P{-}2) \quad \cdots \quad \phi_A(t{+}P,t{+}1)\,B(t) \Big] \tag{1.3}$$

$$C_t := \Big[(C(t))^T \quad (C(t{+}1)\,\phi_A(t{+}1,t))^T \quad \cdots \quad (C(t{+}P{-}1)\,\phi_A(t{+}P{-}1,t))^T \Big]^T \tag{1.4}$$

$$D_t := E_t F_t + G_t,$$

with

$$E_t := \operatorname{diag}\left[\begin{array}{cccc} C(t) & C(t+1) & \cdots & C(t+P-1) \end{array}\right]$$

$$F_t := \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & B(t) \\ \vdots & \vdots & \vdots & & & \iddots & \vdots \\ \vdots & \vdots & \vdots & & \iddots & & \vdots \\ \vdots & \vdots & \vdots & \iddots & & & \vdots \\ 0 & 0 & B(t+P-3) & \cdots & \cdots & \cdots & \phi_A(t+P-2,t+1)\,B(t) \\ 0 & B(t+P-2) & \phi_A(t+P-1,t+P-2)\,B(t+P-3) & \cdots & \cdots & \cdots & \phi_A(t+P-1,t+1)\,B(t) \end{bmatrix}$$

$$G_t := \begin{bmatrix} 0 & 0 & \cdots & 0 & D(t) \\ 0 & 0 & \cdots & D(t+1) & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & D(t+P-2) & \cdots & 0 & 0 \\ D(t+P-1) & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and

$$\phi_A(k,k_0) := A(k-1)\,A(k-2)\cdots A(k_0)\,, \quad k > k_0$$
$$\phi_A(k_0,k_0) := I_n,$$

is the well-known state transition matrix for (1.1).

A straightforward and important result, in [55], is given by the following theorem.

Theorem 1.2.1 [55] *If $(x(k), u(k), y(k))$ is a solution for the P-periodic system $\Sigma_s \equiv (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$, then $(x_t(k), u_t(k), y_t(k))$, defined by*

$$x_t(k) := x(Pk+t)$$

$$u_t(k) := \begin{bmatrix} (u(Pk+t+P-1))^T & (u(Pk+t+P-2))^T & \cdots & (u(Pk+t))^T \end{bmatrix}^T$$

$$y_t(k) := \begin{bmatrix} (y(Pk+t))^T & (y(Pk+t+1))^T & \cdots & (y(Pk+t+P-1))^T \end{bmatrix}^T,$$

is a solution for the time-invariant system $\Sigma_t \equiv (A_t, B_t, C_t, D_t)$, for each $t = 0, \ldots, P–1$. Conversely, if, for each $t = 0, \ldots, P-1$, $(x_t(k), u_t(k), y_t(k))$ is a solution for the time-invariant system Σ_t, with

$$u_t(k) = \begin{bmatrix} (u_t^1(k))^T & (u_t^2(k))^T & \cdots & (u_t^P(k))^T \end{bmatrix}^T$$

$$y_t(k) = \begin{bmatrix} (y_t^1(k))^T & (y_t^2(k))^T & \cdots & (y_t^P(k))^T \end{bmatrix}^T,$$

where each $u_t^i(\cdot), y_t^i(\cdot)$, $i = 1, \ldots, P$, have m and p components, respectively, then $(x(k), u(k), y(k))$, defined by

$$x(k) := x_t(\eta)$$

$$u(k) := \begin{bmatrix} 0 & 0 & \cdots & I_m \end{bmatrix} u_t(\eta)$$

$$y(k) := \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix} y_t(\eta),$$

with $\eta \in \mathbb{Z}$ such that $k = P\eta + t$, is a solution for the P-periodic system Σ_s. $\diamond$

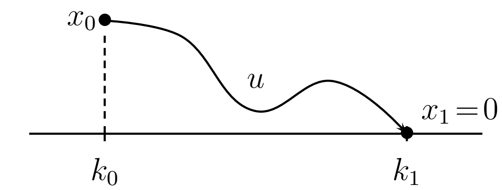
§1.3 Complete state controllability, reachability and trimness

In [10, 55], the structural properties of state controllability and state reachability are introduced following the spirit of the well-known versions for time-invariant systems, yielding more general definitions not depending on the time varying nature of the system.

From here on, since it is clear that we work over the discrete time-axis $\mathbb{Z}$, for simplicity, we will use the interval notation to represent discrete intervals and write, for instance, $[k_1, k_2]$ instead of $[k_1, k_2] \cap \mathbb{Z}$.

Definition 1.3.1 (State controllability)

i) *A state $x_0 \in \mathbb{R}^n$ is called controllable* (*at time k_0*) *if there exist an instant $k_1 \geq k_0$ and an input u* (*defined on $[k_0, k_1 - 1]$*) *that transfers the system from the state $x_0 = x(k_0)$ to $x(k_1) = x_1 = 0 \in \mathbb{R}^n$;*

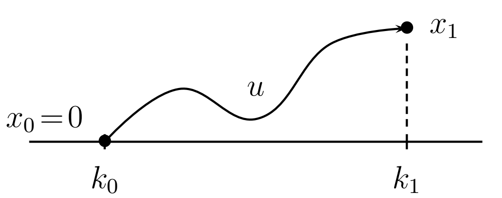


ii) *The system* (1.1) *is called completely state controllable at time k_0 if all the state space $\mathbb{R}^n$ is controllable. Furthermore, if this happens for all $k_0 \in \mathbb{Z}$,* (1.1) *is simply called* completely state controllable.

◇

Definition 1.3.2 (State reachability)

i) *A state $x_1 \in \mathbb{R}^n$ is called reachable* (*at time k_1*) *if there exist an instant $k_0 \leq k_1$ and an input u* (*defined on $[k_0, k_1 - 1]$*) *that transfers the system from the state $x_0 = x(k_0) = 0 \in \mathbb{R}^n$ to $x(k_1) = x_1$;*



ii) *System* (1.1) *is called completely state reachable at time k_1 if all the state space $\mathbb{R}^n$ is reachable. Furthermore, if this happens for all $k_1 \in \mathbb{Z}$,* (1.1) *is simply called* completely state reachable.

◇

Since, as will be seen in the sequel, the characterization of these properties is based on results for time-invariant systems, we will quickly review some relevant facts about state controllability and state reachability of such systems. In what follows the dimension of the state space is assumed to be n.

Theorem 1.3.3 [36, 39, 45, 55, 60, 61] *The following conditions are equivalent:*

i) (A, B, C, D) *is completely state controllable;*

ii) $\operatorname{rank}[\lambda I_n - A \quad B] = n, \ \forall \lambda \in \mathbb{C} \setminus \{0\}$;

iii) $\operatorname{im} A^n \subset \operatorname{im}[B \ \ AB \ \ \cdots \ \ A^{n-1}B]$.

◇

Theorem 1.3.4 [36, 39, 45, 54, 55] *The following conditions are equivalent:*

i) (A, B, C, D) *is completely state reachable;*

ii) $\operatorname{rank}[\lambda I_n - A \quad B] = n, \ \forall \lambda \in \mathbb{C}$;

iii) $\operatorname{rank}[B \ \ AB \ \ \cdots \ \ A^{n-1}B] = n$.

◇

In order to relate state controllability and state reachability we introduce the notion of state trimness, see [60, 61].

Definition 1.3.5 *A state space system is called completely state trim if* $\forall x_0 \in \mathbb{R}^n$, $\forall k_0 \in \mathbb{Z}$, *there exists a system trajectory* (x, u, y) *such that* $x(k_0) = x_0$. ◇

In the time-invariant case, the characterization of trimness, as well as the relevance of this property in relating controllability and reachability, is given in the next two results.

Theorem 1.3.6 [61] *The following conditions are equivalent:*

i) *The time-invariant state space system* (A, B, C, D) *is completely state trim;*

ii) $\operatorname{rank}[A \quad B] = n$.

◇

Combining Theorems 1.3.3, 1.3.4 and 1.3.6, we can conclude that the following holds.

Theorem 1.3.7 *A time-invariant state space system is completely state reachable if and only if it is completely state controllable and completely state trim.* $\diamond$

Example 1.3.8 *The time-invariant system with dynamics*

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \quad k \in \mathbb{Z},$$

although completely state controllable is not completely state reachable since it is not completely state trim as

$$\operatorname{rank} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \right] = 1.$$

$\diamond$

Concerning the P-periodic case, the characterization of trimness can also be achieved in terms of a system's time-invariant decomposition.

Theorem 1.3.9 *The following conditions are equivalent:*

i) *The P-periodic state space system* (1.1) *is completely state trim;*

ii) *The associated P time-invariant systems Σ_t,*

$$(A_t, B_t, C_t, D_t), \quad t = 0, \ldots, P-1,$$

are all completely state trim;

iii) $\operatorname{rank}[A_t \quad B_t] = n, \quad t = 0, \ldots, P-1.$

Proof. The equivalence between conditions ii) and iii) follows immediately from Theorem 1.3.6 since the systems Σ_t are time-invariant. The equivalence between conditions i) and ii) can be easily obtained taking into account of how the trajectories of the P-periodic system are related to the ones of the P time-invariant systems. Indeed, if (1.1) is not completely state trim at some time instant $k_0 = P\eta_0 + t$, then some time-invariant system Σ_t is not completely state trim at time η_0. Reciprocally, if some Σ_t is not completely state trim at time η_0, then (1.1) is not completely state trim at time $k_0 = P\eta_0 + t$. $\square$

In [55] several results are obtained concerning the characterization of the state controllability and state reachability of Σ_s based on the state controllability and state reachability of each system Σ_t of its time-invariant decomposition and known results for the time-invariant case.

Theorem 1.3.10 [55] *The P-periodic state space system Σ_s is completely state controllable if and only if any of the two following equivalent conditions holds:*

i) *All the P time-invariant systems Σ_t are completely state controllable;*

ii) *At least one of the P time-invariant systems Σ_t is completely state controllable.*

$\diamond$

Theorem 1.3.11 [55] *The P-periodic state space system Σ_s is completely state reachable if and only if all the P time-invariant systems Σ_t are completely state reachable.* $\diamond$

By combining Theorems 1.3.10, 1.3.11 and 1.3.7, we may easily state the following result.

Theorem 1.3.12 *The P-periodic state space system Σ_s is completely state reachable if and only if all the P time-invariant systems Σ_t are completely state trim and at least one of them is completely state controllable.* $\diamond$

Indeed, the periodic state space system Σ_s is completely state reachable if and only if all the corresponding systems Σ_t in the time-invariant formulation are completely state reachable. By Theorem 1.3.7, this means that all the systems Σ_t are completely state controllable and completely state trim, which by Theorem 1.3.10, is in turn equivalent of saying that all the systems Σ_t are completely state trim and at least one of them is completely state controllable.

Taking Theorems 1.3.9, 1.3.10 and 1.3.12 into account we obtain a similar result to Theorem 1.3.7 within the P-periodic case.

Theorem 1.3.13 *A P-periodic state space system is completely state reachable if and only if it is completely state controllable and completely state trim.* $\diamond$

§1.4 Complete state reconstructibility and observability

In [10, 55], the structural properties of state reconstructibility and state observability are introduced following again the spirit of the well-known versions for time-invariant systems.

Definition 1.4.1 (State reconstructibility)

i) *A state* $x_1 \in \mathbb{R}^n$ *is called unreconstructible* (*at time* k_1) *if for all* $k_0 \leq k_1$*, there exists* $x_0 = x(k_0) \in \mathbb{R}^n$ *such that*

$$y(k) = C(k)\,\phi_A(k, k_0)\,x_0 = 0, \quad k \in [k_0, k_1 - 1],$$

with $x_1 = x(k_1)$*;*

ii) *The system* (1.1) *is called completely state reconstructible at time* k_1 *if the only state* x_1 *that is unreconstructible is the zero state, i.e.,* $x_1 = 0 \in \mathbb{R}^n$*. If this happens for all* $k_1 \in \mathbb{Z}$*,* (1.1) *is simply called* completely state reconstructible.

◇

Definition 1.4.2 (State observability)

i) *A state* $x_0 \in \mathbb{R}^n$ *is called unobservable* (*at time* k_0) *if for all* $k_1 \geq k_0$

$$y(k) = C(k)\,\phi_A(k, k_0)\,x_0 = 0, \quad k \in [k_0, k_1 - 1]$$

i.e., if the zero input response of the system is zero for every $k \geq k_0$*;*

ii) *The system* (1.1) *is called completely state observable at time* k_0 *if the only state* x_0 *that is unobservable is the zero state, i.e.,* $x_0 = 0 \in \mathbb{R}^n$*. If this happens for all* $k_0 \in \mathbb{Z}$*,* (1.1) *is simply called* completely state observable.

◇

We next present some well-known facts about the state reconstructibility and state observability of time-invariant systems, that will be relevant for the characterization of these properties in the periodic case. For this purpose, let (A, B, C, D) be a time-invariant state space system.

Theorem 1.4.3 [36, 39, 45, 55] *The following conditions are equivalent:*

i) (A, B, C, D) *is completely state reconstructible;*

ii) $\operatorname{rank} \begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \setminus \{0\};$

iii) $\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \subset \ker A^n.$

$\diamond$

Theorem 1.4.4 [36, 39, 45, 54, 55, 61] *The following conditions are equivalent:*

i) (A, B, C, D) *is completely state observable;*

ii) $\operatorname{rank} \begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C};$

iii) $\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$

$\diamond$

Comparing these results with Theorems 1.3.3 and 1.3.4, it is clear that complete state reconstructibility and complete state observability are dual with respect to complete state controllability and complete state reachability, in the sense that (A, B, C, D) is complete state reconstructible/observable if and only if $\left(A^T, C^T, B^T, D^T\right)$ is complete state controllable/reachable.

In [55] several results are obtained concerning the characterization of the state reconstructibility and state observability of Σ_s based on the state reconstructibility and state observability of each of the associated time-invariant systems Σ_t and on known results for the time-invariant case.

Theorem 1.4.5 [55] *The P-periodic state space system Σ_s is completely state reconstructible if and only if all the P time-invariant systems Σ_t are completely state reconstructible.* $\diamond$

Theorem 1.4.6 [55] *The P-periodic state space system Σ_s is completely state observable if and only if all the P time-invariant systems Σ_t are completely state observable.* $\diamond$

As happens in the time-invariant case, complete state reconstructibility and complete state observability can be regarded as dual properties with respect to controllability and reachability. However, the notion of duality for the P-periodic case requires a more careful definition than in the time-invariant case.

The dual of a P-periodic state space system $\Sigma_s = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is defined as

$$\widetilde{\Sigma}_s = \left(\widetilde{A}(\cdot), \widetilde{B}(\cdot), \widetilde{C}(\cdot), \widetilde{D}(\cdot)\right),$$

where $\widetilde{A}(k) = (A(-k))^T$, $\widetilde{B}(k) = (C(-k))^T$, $\widetilde{C}(k) = (B(-k))^T$, $\widetilde{D}(k) = (D(-k))^T$. As shown in [55], if $\Sigma_t = (A_t, B_t, C_t, D_t)$, $t = 0, \ldots, P-1$, is the time-invariant formulation of Σ_s, then the time-invariant formulation of $\widetilde{\Sigma}_s$ is such that

$$\widetilde{\Sigma}_p = \left(\widetilde{A}_p, \widetilde{B}_p, \widetilde{C}_p, \widetilde{D}_p\right) = \left(A_t^T, C_t^T, B_t^T, D_t^T\right), \quad p = 1 - t.$$

In view of Theorems 1.4.5 and 1.4.6, this allows us to conclude that Σ_s is completely state reconstructible/observable if and only if $\widetilde{\Sigma}_s$ is completely state controllable/reachable. The relation between complete state reconstructibility and complete state observability can be established from this correspondence, together with the considerations made in Section 1.3.

§1.5 Conclusion

The main core of this chapter is devoted to the presentation of results obtained in Ana Urbano's PhD thesis, [55], namely the time-invariant dynamical decomposition of a periodic state space system and some results linking its structural properties with the properties of the associated time-invariant systems. This has been completed with the study of the relation between the properties of complete state controllability and complete state reachability (complete state reconstructibility and complete state observability), which has required the characterization of the notion of trimness, introduced in [60], in the P-periodic context.

"Science is organized knowledge. Wisdom is organized life."

— Immanuel Kant

"It is unwise to be too sure of one's own wisdom. It is healthy to be reminded that the strongest might weaken and the wisest might err."

— Mahatma Gandhi

2 Time-invariant behaviors

This chapter contains an overview of the background material concerning time-invariant behaviors in order to more easily establish the connection between periodic systems and their associated time-invariant formulations, to be introduced later on. Some of these subjects are well-known within the behavioral approach while others have been developed during our research.

§2.1 Introduction

In the behavioral framework, see [60, 61], a dynamical system Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ as the time set, $\mathbb{W}$ as the signal space and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}} := \{w : \mathbb{T} \to \mathbb{W}\}$ as the behavior. The behavior $\mathfrak{B}$ is what characterizes the phenomenon described by the system Σ, since it consists of all the signal evolutions (system trajectories) that are compatible with the laws of that phenomenon. In this thesis we shall be concerned with the discrete-time case, that is, $\mathbb{T} = \mathbb{Z}$, assuming furthermore that the signal space is $\mathbb{W} = \mathbb{R}^q$, with $q \in \mathbb{N}$. Thus, $\mathbb{W}^{\mathbb{T}} = (\mathbb{R}^q)^{\mathbb{Z}}$, i.e., the system trajectories are $\mathbb{R}^q$-valued sequences over $\mathbb{Z}$.

In this chapter we will define some fundamental properties of dynamical systems under the behavioral scope.

Definition 2.1.1 [60, 61] *A dynamical system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *is said to be* linear *if* $\mathbb{W}$ *is a vector space* (*over a field* $\mathbb{F}$) *and* $\mathfrak{B}$ *is a linear subspace of* $\mathbb{W}^{\mathbb{T}}$ (*which is obviously a vector space when equipped with the usual operations of pointwise addition and multiplication by a scalar*). $\diamond$

Thus, linear systems obey the *superposition principle* in its simplest form:

$$w_1(\cdot),\ w_2(\cdot) \in \mathfrak{B}; \quad \alpha, \beta \in \mathbb{F} \quad \Rightarrow \quad \alpha w_1(\cdot) + \beta w_2(\cdot) \in \mathfrak{B}.$$

In order to introduce the notion of time-invariance it is essential to define first how to *shift-in-time* the signals w.

Given $\tau \in \mathbb{Z}$, we define the τ-shift as $\sigma^\tau : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}}$, such that:

$$(\sigma^\tau w)(k) := w(k + \tau);$$

σ^τ is called the *backward τ-shift* in case $\tau \in \mathbb{Z}_+$, and the *forward τ-shift* in case $\tau \in \mathbb{Z}_-$.

Definition 2.1.2 [60, 61] *A dynamical system* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *is said to be* time-invariant *if* $\sigma(\mathfrak{B}) = \mathfrak{B}$. $\diamond$

Thus, time-invariance is nothing else than invariance with respect to the time shift σ.

Another important notion is the completeness of a behavior $\mathfrak{B}$, meaning, roughly speaking, that it is possible to check whether a trajectory $w \in (\mathbb{R}^q)^{\mathbb{Z}}$ belongs to $\mathfrak{B}$, by checking what happens in the set $\mathfrak{I}$ of finite intervals of $\mathbb{Z}$. Given $I \in \mathfrak{I}$, denote by $\mathfrak{B}\big|_I$ the set of all the restrictions of the system trajectories to the time interval I, that is,

$$\mathfrak{B}\Big|_I = \left\{ w\Big|_I :\ w \in \mathfrak{B} \right\}.$$

Definition 2.1.3 [60, 61] *A dynamical system* $\Sigma = (\mathbb{Z}, \mathbb{W}, \mathfrak{B})$ *is said to be* complete *if*

$$\left(\forall I \in \mathfrak{I}, \quad w\Big|_I \in \mathfrak{B}\Big|_I \right) \Leftrightarrow w \in \mathfrak{B}.$$

$\diamond$

§2.2 Kernel representations

A crucial issue is the representation of the behavior of a system by means of mathematical equations. It turns out that all the discrete-time dynamical systems that are linear, time-invariant and complete, allow a special type of mathematical description known as kernel representation, see [60, 61].

Theorem 2.2.1 [60,61] *Let* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *be a dynamical system. The following are equivalent:*

i) Σ *is linear, time-invariant and complete;*

ii) $\exists R(\xi, \xi^{-1}) \in \mathbb{R}^{\bullet \times q}[\xi, \xi^{-1}]$ *such that*

$$\mathfrak{B} = \ker R\left(\sigma, \sigma^{-1}\right) := \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}} : \ R\left(\sigma, \sigma^{-1}\right) w = 0 \right\},$$

where $\mathbb{R}^{\bullet \times q}[\xi, \xi^{-1}]$ *denotes the set of* $\bullet \times q$ *matrices with entries in* $\mathbb{R}[\xi, \xi^{-1}]$*, the ring of Laurent-polynomials in the indeterminate* ξ*.*

$\diamond$

Condition ii), in Theorem 2.2.1, means that there exists a certain Laurent-polynomial matrix

$$R\left(\xi, \xi^{-1}\right) = R^{-M}\xi^{-M} + \cdots + R^0 + \cdots + R^N \xi^N,$$

with $N, M \in \mathbb{Z}_+$, such that the trajectories $w \in \mathfrak{B}$ are the elements of $(\mathbb{R}^q)^{\mathbb{Z}}$ which constitute a solution of the linear constant coefficient matrix difference equation

$$\begin{aligned} R^{-M} w(k-M) + \cdots + R^{-1} w(k-1) + R^0 w(k) \\ + R^1 w(k+1) + \cdots + R^N w(k+N) = 0, \forall k \in \mathbb{Z}. \end{aligned}$$

This matrix $R(\xi, \xi^{-1})$ is called a *kernel representation (KR)* matrix of $\mathfrak{B}$. A behavior that allows a KR is called *kernel behavior*. From here on, since we shall only consider kernel behaviors, we will drop the term “kernel” and simply refer to “*behaviors*”.

Remark 2.2.2 *In order to specify that a system* Σ*, or the corresponding behavior* $\mathfrak{B}$*, is described by certain equations we use the notation* $\Sigma \sim$ *or* $\mathfrak{B} \sim$ *followed by these equations. For instance, if* $\mathfrak{B} = \ker R(\sigma, \sigma^{-1})$*, we write*

$$\mathfrak{B} \ \sim \ \left(R\left(\sigma, \sigma^{-1}\right) w\right)(k) = 0, \quad k \in \mathbb{Z}$$

or simply

$$\mathfrak{B} \ \sim \ Rw = 0.$$

$\diamond$

Whereas a KR matrix uniquely defines the associated behavior, the same behavior may allow different kernel representations. Two KR matrices $R_1(\xi, \xi^{-1})$ and $R_2(\xi, \xi^{-1})$ are said to be *equivalent* if the corresponding behavior is the same, i.e., if $\ker R_1(\sigma, \sigma^{-1}) = \ker R_2(\sigma, \sigma^{-1})$. The results presented below yield a characterization of equivalent kernel representations.

Theorem 2.2.3 [59] *Consider the two time-invariant behaviors*

$$\mathfrak{H} = \ker H\left(\sigma, \sigma^{-1}\right) \quad \text{and} \quad \mathfrak{H}' = \ker H'\left(\sigma, \sigma^{-1}\right).$$

Then $\mathfrak{H} \subseteq \mathfrak{H}'$ if and only if there exists a Laurent-polynomial matrix $V\left(\xi, \xi^{-1}\right)$ such that

$$H'\left(\xi, \xi^{-1}\right) = V\left(\xi, \xi^{-1}\right) H\left(\xi, \xi^{-1}\right).$$

◇

An immediate consequence of this latter result is that $\mathfrak{H} = \mathfrak{H}'$ if and only if there exist two Laurent-polynomial matrices, $V\left(\xi, \xi^{-1}\right)$ and $V'\left(\xi, \xi^{-1}\right)$, such that

$$H'\left(\xi, \xi^{-1}\right) = V\left(\xi, \xi^{-1}\right) H\left(\xi, \xi^{-1}\right)$$

and

$$H\left(\xi, \xi^{-1}\right) = V'\left(\xi, \xi^{-1}\right) H'\left(\xi, \xi^{-1}\right).$$

Furthermore, it is shown in [59] that if both matrices H and H' have the same number of rows, then $\mathfrak{H} = \mathfrak{H}'$ if and only if there exists a unimodular matrix $U\left(\xi, \xi^{-1}\right)$ such that

$$H'\left(\xi, \xi^{-1}\right) = U\left(\xi, \xi^{-1}\right) H\left(\xi, \xi^{-1}\right).$$

The concept of *minimality* plays an important role in the study of kernel representations. As formulated in the next definition, minimality simply corresponds to a minimal number of scalar equations (rows of the kernel representation matrix).

Definition 2.2.4 [60, 61] *A KR matrix $R \in \mathbb{R}^{g \times q}\left[\xi, \xi^{-1}\right]$ of a time-invariant system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is said to be* minimal *if for any other representation $R' \in \mathbb{R}^{g' \times q}\left[\xi, \xi^{-1}\right]$ of Σ, there holds $g \leq g'$.* ◇

Proposition 2.2.5 [60, 61] *Every behavior admits a minimal representation. Moreover, every minimal KR matrix R has full row rank (frr).* ◇

§2.3 Latent variable and image representations

In addition to kernel representations, which only involve the system variables, it is sometimes useful also to consider the descriptions with other auxiliary variables. This is what happens, for instance, when models are derived from first principles, or when one is interested in obtaining simplified descriptions, such as state space models. In the behavioral setting, those auxiliary variables are usually called latent variables as opposed to the system variables which are said to be manifest.

Definition 2.3.1 [60, 61] *A system* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *is said to have a* latent variable representation (LVR) *if its behavior consists of all the manifest trajectories* w *that satisfy, together with some latent variable trajectory* $v \in \left(\mathbb{R}^\ell\right)^{\mathbb{Z}}$, *the following set of equations:*

$$\left(R\left(\sigma, \sigma^{-1}\right) w\right)(k) = \left(M\left(\sigma, \sigma^{-1}\right) v\right)(k), \quad k \in \mathbb{Z} \tag{2.1}$$

and where $R(\xi, \xi^{-1}) \in \mathbb{R}^{g \times q}[\xi, \xi^{-1}]$, $M(\xi, \xi^{-1}) \in \mathbb{R}^{g \times \ell}[\xi, \xi^{-1}]$, *i.e.,*

$$\mathfrak{B} = \left\{w \in (\mathbb{R}^q)^{\mathbb{Z}} \,|\, \exists v \in \left(\mathbb{R}^\ell\right)^{\mathbb{Z}} \text{ s.t. } (2.1) \text{ holds}\right\}.$$

The LVR (2.1) *is denoted by* (R, M). $\diamond$

A special case of latent variable representation occurs when $R = I_q$. In this case equation (2.1) becomes

$$w(k) = \left(M\left(\sigma, \sigma^{-1}\right) v\right)(k), \quad k \in \mathbb{Z}.$$

Definition 2.3.2 [60, 61] *A system* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *is said to have an* image representation (IR) *if its behavior can be written as*

$$\mathfrak{B} = \left\{w \in (\mathbb{R}^q)^{\mathbb{Z}} \,|\, \exists v \in \left(\mathbb{R}^\ell\right)^{\mathbb{Z}} \text{ s.t. } w(k) = \left(M\left(\sigma, \sigma^{-1}\right) v\right)(k)\right\}.$$

In this case M *is said to be an* IR *matrix, of* $\mathfrak{B}$. $\diamond$

§2.3.1 Latent variable elimination

The manifest behavior described by a LVR can also be described by a KR. This means that the latent variables v can be eliminated from a LVR (R, M), $Rw = Mv$, and the restrictions imposed on the system variables w can be written in terms of the variable w alone. As stated in the next theorem, the latent variable elimination procedure is based on the application, to both sides of (2.1), of a shift-operator corresponding to a minimal left annihilator of the matrix M. A Laurent-polynomial matrix $L(\xi, \xi^{-1})$ is said to be a minimal left annihilator (MLA) of $M(\xi, \xi^{-1})$ if $L(\xi, \xi^{-1})\, M(\xi, \xi^{-1}) = 0$ and, moreover, whenever $K(\xi,\xi^{-1}) M(\xi,\xi^{-1}) = 0$ then $K(\xi,\xi^{-1}) = X(\xi,\xi^{-1}) L(\xi,\xi^{-1})$, for some $X(\xi,\xi^{-1})$, see [53,64]. This means that the rows of L generate all the annihilators of M.

Theorem 2.3.3 (latent variable elimination) [51,52,61] *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a dynamical system. Then $\mathfrak{B}$ has a LVR (R, M) if and only if it has a KR R^*. Given (R, M), R^* may be obtained as*

$$R^*\left(\xi,\xi^{-1}\right) := H\left(\xi,\xi^{-1}\right) R\left(\xi,\xi^{-1}\right),$$

where $H(\xi,\xi^{-1})$ is a minimal left annihilator (MLA) *of the matrix $M(\xi,\xi^{-1})$.* ◇

Note that all these considerations can be easily particularized to the case of image behaviors.

Since IR is a particular type of LVR, it is natural to expect that it represents a more restricted class of systems. As we shall see, in the next section, systems with a IR are precisely those which are controllable.

§2.4 Controllability, autonomy and free variables

The concept of controllability undoubtedly plays a central role within the systems theory. Roughly speaking, behavioral controllability means that it is possible to concatenate, in finite time, the past and the future of any two system trajectories.

Definition 2.4.1 [60,61] *A system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ and its behavior $\mathfrak{B}$ are said to be controllable, or to have a controllable behavior, if for all $w_1, w_2 \in \mathfrak{B}$ and all $k_0 \in \mathbb{Z}$, there exist $w \in \mathfrak{B}$ and $k_1 \geq 0$ such that*

$$w(k) = \begin{cases} w_1(k), & k \leq k_0 \\ w_2(k), & k > k_0 + k_1, \end{cases}$$

holds. ◇

Note that, due to linearity, the Definition 2.4.1 of controllability is equivalent to the possibility of driving every trajectory to the zero trajectory.

Lemma 2.4.2 *A behavior* $\mathfrak{B}$ *is* (behaviorally) controllable *if and only if for all* $w' \in \mathfrak{B}$ *and* $k_0 \in \mathbb{Z}$*, there exist* $k_1 \geq 0$ *and* $w \in \mathfrak{B}$ *such that* $w(k) = w'(k)$*, for* $k \leq k_0$*, and* $w(k) = 0$*, for* $k > k_0 + k_1$*.*

Proof. The *only if* part is obvious, since it is enough to take, in Definition 2.4.1, $w_2 = 0$. In order to verify the *if* part take any two trajectories in the behavior, say w_1 and w_2, then their difference $w_1 - w_2$ may be concatenated in finite time with the zero trajectory. Consequently this overall trajectory, that is, the trajectory

$$\begin{cases} w_1(k) - w_2(k), & k \leq k_0 \\ 0, & k > k_0 + k_1, \end{cases}$$

belongs to the behavior. Therefore by adding w_2 to this latter trajectory we get the desired controllability requirement. □

In the sequel we give a characterization for the controllability of behaviors.

Theorem 2.4.3 [60, 61] *Let* $\mathfrak{B} = \ker R(\sigma, \sigma^{-1})$*, with* $R(\xi, \xi^{-1}) \in \mathbb{R}^{g \times q}[\xi, \xi^{-1}]$*. Then the following conditions are equivalent:*

i) $\mathfrak{B}$ *is controllable;*

ii) $\operatorname{rank} R(\lambda, \lambda^{-1})$ *is constant for all* $\lambda \in \mathbb{C} \setminus \{0\}$*;*

iii) *there exists an image representation (IR) for* $\mathfrak{B}$*.*

◇

Thus, controllable behaviors are exactly those that allow an image representation and, therefore, are also *image behaviors*. Based on this fact, and using arguments involving the degree of the IR matrix, it is not difficult to conclude that the *control time*, i.e., the time lag k_1 in Definition 2.4.1 (which *a priori* may depend on the pair of trajectories to be concatenated) can be taken to be fixed for each behavior.

Proposition 2.4.4 *A behavior* $\mathfrak{B}$ *is controllable if and only if there exists* $L \in \mathbb{N}$ *such that, for all* $w_1, w_2 \in \mathfrak{B}$ *and all* $k_0 \in \mathbb{Z}$*, there exists* $w \in \mathfrak{B}$ *satisfying*

$$w(k) = \begin{cases} w_1(k), & k \leq k_0 \\ w_2(k), & k \geq k_0 + L. \end{cases}$$

Proof. Assume that $\mathfrak{B}$ is controllable. Then, according to Theorem 2.4.3, there exists a Laurent-polynomial matrix M such that $\mathfrak{B} = \operatorname{im} M(\sigma, \sigma^{-1})$. It is not difficult to show (by, if necessary, redefining the latent variable) that, in this case, there also exists a polynomial matrix

$$\widetilde{M}(\xi) = M_\alpha \xi^\alpha + \cdots + M_0, \quad \text{with } \alpha \geq 0,$$

such that

$$\mathfrak{B} = \operatorname{im} \widetilde{M}(\sigma).$$

Let $w_1, w_2 \in \mathfrak{B}$. Then there exist $v_1, v_2 \in \left(\mathbb{R}^\ell\right)^\mathbb{Z}$ such that $w_1 = \widetilde{M} v_1$ and $w_2 = \widetilde{M} v_2$. Define now $L := \alpha + 1$. Given any $k_0 \in \mathbb{Z}$, let $v \in \left(\mathbb{R}^\ell\right)^\mathbb{Z}$ be such that

$$v(k) = \begin{cases} v_1(k), & k \leq k_0 + \alpha \\ v_2(k), & k > k_0 + \alpha. \end{cases}$$

It can be easily ascertained that $w = \widetilde{M} v$ is a trajectory in the behavior such that

$$w(k) = \begin{cases} w_1(k), & k \leq k_0 \\ w_2(k), & k \geq k_0 + L. \end{cases}$$

The reciprocal implication is obvious. □

If the condition of Proposition 2.4.4 holds for some $L \in \mathbb{N}$, we say that $\mathfrak{B}$ is *controllable with control time* L.

At the extreme opposite of controllability stands autonomy, which is the impossibility of connecting a system trajectory with another different one, meaning that every trajectory in an autonomous behavior is uniquely determined by its past.

Definition 2.4.5 [60,61] *A system* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *is said to be autonomous if for all* $k_0 \in \mathbb{Z}$ *and all* $w_1, w_2 \in \mathfrak{B}$

$$w_1(k) = w_2(k) \text{ for } k < k_0 \quad \Rightarrow \quad w_1 = w_2.$$

◇

Note that Definitions 2.4.1 and 2.4.5 are valid regardless of whether the system is time-invariant or not, and will be also used in Chapter 4 for periodic systems. Note also that, for time-invariant systems, k_0 in Definition 2.4.5 can be replaced by 0. Moreover, due to linearity, we may state that $\mathfrak{B}$ is autonomous if and only if

$$w(k) = 0,\ k \leq 0 \quad \Rightarrow \quad w(k) = 0,\ \forall k \in \mathbb{Z}.$$

In the time-invariant setting the concept of autonomy is strictly connected with the concept of free variables. Given a behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$, a component w_i of the system variable is said to be *free* if for all $\alpha \in \mathbb{R}^{\mathbb{Z}}$ there exists a trajectory $w^* \in \mathfrak{B}$ such that $w_i^*(k) = \alpha(k),\ k \in \mathbb{Z}$, i.e., w_i is not restricted by the system laws.

This is put into evidence in the following characterization of autonomy for behaviors.

Theorem 2.4.6 [60, 61] *Let* $\mathfrak{B} = \ker R(\sigma, \sigma^{-1})$, *with* $R(\xi, \xi^{-1}) \in \mathbb{R}^{g\times q}[\xi, \xi^{-1}]$. *Then the following conditions are equivalent:*

i) $\mathfrak{B}$ *is autonomous;*

ii) R *has full column rank;*

iii) $\mathfrak{B}$ *has no free variables.*

$\diamond$

Note that this result also allows us to conclude that a non-trivial time-invariant controllable behavior must have free variables. The interest of this obvious remark will become clear later when we consider the periodic case.

Similar to what happens for state space systems, every behavior $\mathfrak{B}$ can be decomposed as the direct sum of an autonomous sub-behavior with a controllable sub-behavior, more concretely the following result holds true.

Theorem 2.4.7 [61] *Let* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *be a system with behavior* $\mathfrak{B}$. *Then:*

i) *there exist an autonomous sub-behavior,* $\mathfrak{B}^{\mathrm{a}}$, *of* $\mathfrak{B}$, *and a controllable sub-behavior,* $\mathfrak{B}^{\mathrm{c}}$, *of* $\mathfrak{B}$, *such that* $\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}$;

ii) *if* $\mathfrak{B}_1^{\mathrm{a}}, \mathfrak{B}_2^{\mathrm{a}}$, *autonomous, and* $\mathfrak{B}_1^{\mathrm{c}}, \mathfrak{B}_2^{\mathrm{c}}$, *controllable, are sub-behaviors of* $\mathfrak{B}$ *such that* $\mathfrak{B} = \mathfrak{B}_1^{\mathrm{a}} \oplus \mathfrak{B}_1^{\mathrm{c}} = \mathfrak{B}_2^{\mathrm{a}} \oplus \mathfrak{B}_2^{\mathrm{c}}$, *then* $\mathfrak{B}_1^{\mathrm{c}} = \mathfrak{B}_2^{\mathrm{c}}$.

$\diamond$

The sub-behavior $\mathfrak{B}^{\mathrm{c}}$ in this decomposition is called the *controllable part* of $\mathfrak{B}$.

§2.4.1 Controllability of time-invariant state space systems

In this subsection we present a comparison between the classical concept of complete state controllability as introduced in Section 1.3 and the concept of behavioral controllability, now, applied to time-invariant state space systems. A state space system will be behaviorally controllable if the concatenation property of Definition 2.4.1 holds for any two trajectories $w_1 = (x_1, u_1)$ and $w_2 = (x_2, u_2)$.

As stated in the next theorem these properties appear to coincide.

Theorem 2.4.8 [52] *Let* $\Sigma = (\mathbb{Z}, \mathbb{R}^q \simeq \mathbb{R}^n \times \mathbb{R}^m, \mathfrak{B})$ *be a time-invariant state space system, i.e., a system with signals* $w = (x, u)$ *and behavior described by*

$$\mathfrak{B} = \ker \begin{bmatrix} \sigma I_n - A & B \end{bmatrix}.$$

Then the following conditions are equivalent:

i) $\mathfrak{B}$ *is completely state controllable;*

ii) $\mathfrak{B}$ *is (behaviorally) controllable;*

iii) $\operatorname{rank}[\lambda I_n - A \quad B] = n, \ \forall \lambda \in \mathbb{C} \setminus \{0\}$.

$\diamond$

Note that condition iii) of Theorem 2.4.8 coincides with the given characterization previously stated in Theorem 1.3.3 for the complete state controllability of a time-invariant system.

§2.5 Reconstructibility and forward-observability

Similar to controllability and reachability, the properties of reconstructibility and observability also play a central role in systems theory. These properties are related with the possibility of obtaining information about some components of the system variable, which cannot be directly measured, based on the knowledge of the other components, which are assumed to be available for measurement.

Definition 2.5.1 *Let $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}} \simeq (\mathbb{R}^{q_1} \times \mathbb{R}^{q_2})^{\mathbb{Z}}$ be a behavior whose system variable w is partitioned as $w = (w_1, w_2)$. Given $\delta \geq 0$, we say that w_2 is* δ-reconstructible *from w_1 if*

$$\left\{ w_1\Big|_{[k_0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ w_2\Big|_{[k_0+\delta,+\infty)} \equiv 0 \right\}, \quad \forall k_0 \in \mathbb{Z}. \tag{2.2}$$

Moreover, w_2 is said to be reconstructible *from w_1 if it is δ-reconstructible from w_1 for some $\delta \geq 0$. In particular, w_2 is said to be* forward-observable *from w_1 if it is 0-reconstructible from w_1, i.e., if*

$$\left\{ w_1\Big|_{[k_0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ w_2\Big|_{[k_0,+\infty)} \equiv 0 \right\}, \quad \forall k_0 \in \mathbb{Z}. \tag{2.3}$$

$\diamond$

Note that this definition does not depend on the time-invariant nature of the systems under consideration. Therefore it will be used later on in the context of the periodic case.

Example 2.5.2 *Consider a system $\Sigma = (\mathbb{Z}, \mathbb{R}^2, \mathfrak{B})$ with variables (w_1, w_2), whose behavior $\mathfrak{B}$ is described by*

$$\sigma w_2 = w_1,$$

i.e.,

$$w_2(k) = w_1(k-1), \quad \forall k \in \mathbb{Z}.$$

Clearly w_2 is 1-reconstructible from w_1, since

$$w_1(k) = 0, \quad k \geq k_0$$

implies

$$w_2(k) = w_1(k-1) = 0, \quad k \geq k_0 + 1.$$

It is also simple to see that w_2 is not forward-observable from w_1. Indeed, if $w_1(-1) = 1$ and $w_1(k) = 0$ for $k \neq -1$, we have that $w_2(0) = 1$ and $w_2(k) = 0$ for $k \neq 0$. Thus

$$w_1\Big|_{[0,+\infty)} = 0, \quad \textit{but} \quad w_2\Big|_{[0,+\infty)} \neq 0.$$

However, w_1 is forward-observable from w_2, as can easily be ascertained. $\diamond$

Note that, due to time-invariance, the δ-reconstructibility condition (2.2) in Definition 2.5.1 can be replaced by

$$\left\{ w_1\Big|_{[0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ w_2\Big|_{[\delta,+\infty)} \equiv 0 \right\}, \tag{2.4}$$

whereas the forward-observability condition (2.3) can be replaced by

$$\left\{ w_1\Big|_{[0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ w_2\Big|_{[0,+\infty)} \equiv 0 \right\}.$$

This agrees with the definitions of reconstructibility and observability given in [56], for discrete-time systems over $\mathbb{Z}_+$, but not with the definition of observability given in [60, 61], according to which w_2 is said to be observable from w_1 if

$$\left\{ w_1(k) = 0,\ \forall k \in \mathbb{Z} \right\} \Rightarrow \left\{ w_2(k) = 0,\ \forall k \in \mathbb{Z} \right\}. \tag{2.5}$$

In the sequel, to avoid confusion we refer to this property as *Willems-observability*.

Proposition 2.5.3 *Let $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ be a time-invariant behavior whose system variable w is partitioned as $w = (w_1, w_2)$. Then w_2 is* Willems-observable *from w_1 if it is* reconstructible *from w_1.*

Proof. Assume that w_2 is δ-reconstructible from w_1, for some $\delta \geq 0$. Consider a trajectory $(w_1', w_2') \in \mathfrak{B}$ with $w_1'\Big|_{\mathbb{Z}} = 0$. In particular,

$$w_1'\Big|_{[k_0,+\infty)} \equiv 0, \quad \forall k_0 \in \mathbb{Z},$$

and therefore

$$w_2'\Big|_{[k_0+\delta,+\infty)} \equiv 0, \quad \forall k_0 \in \mathbb{Z}.$$

This clearly implies that $w_2'\Big|_{\mathbb{Z}} \equiv 0$, allowing to conclude that reconstructibility implies Willems-observability. □

As we shall see in the following subsection, the reciprocal of these result is also valid.

§2.5.1 Reconstructibility and forward-observability characterization

In this subsection we characterize reconstructibility and forward-observability by means of rank conditions.

Theorem 2.5.4 *Consider the dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{q_1+q_2}, \mathfrak{B})$ described by*

$$\mathfrak{B} := \left\{ (w_1, w_2) \in \left(\mathbb{R}^{q_1+q_2}\right)^{\mathbb{Z}} \mid \left(R_2\left(\sigma, \sigma^{-1}\right) w_2\right)(k) = \left(R_1\left(\sigma, \sigma^{-1}\right) w_1\right)(k),\ k \in \mathbb{Z} \right\},$$

with $R_2(\xi, \xi^{-1}) \in \mathbb{R}^{g \times q_2}[\xi, \xi^{-1}]$, $R_1(\xi, \xi^{-1}) \in \mathbb{R}^{g \times q_1}[\xi, \xi^{-1}]$. Then,

i) *w_2 is reconstructible from w_1 if and only if*

$$\operatorname{rank} R_2\left(\lambda, \lambda^{-1}\right) = q_2, \quad \forall \lambda \in \mathbb{C} \setminus \{0\}; \tag{2.6}$$

ii) *w_2 is forward-observable from w_1 if and only if there exist $\widetilde{R}_2(\xi) \in \mathbb{R}^{g \times q_2}[\xi]$ and $\widetilde{R}_1(\xi) \in \mathbb{R}^{g \times q_1}[\xi]$ such that $\mathfrak{B}$ is described by $\widetilde{R}_2(\sigma) w_2 = \widetilde{R}_1(\sigma) w_1$, with*

$$\operatorname{rank} \widetilde{R}_2(\lambda) = q_2, \quad \forall \lambda \in \mathbb{C}. \tag{2.7}$$

Proof.

i) Assume that (2.6) holds. Then, there exists a matrix $U(\xi, \xi^{-1}) \in \mathbb{R}^{g \times g}[\xi, \xi^{-1}]$, which is unimodular over $\mathbb{R}[\xi, \xi^{-1}]$, such that [52],

$$U\left(\xi, \xi^{-1}\right) R_2\left(\xi, \xi^{-1}\right) = \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix}.$$

Thus (leaving out σ and σ^{-1} in the notation, for simplicity),

$$\begin{aligned} R_2 w_2 = R_1 w_1 &\Leftrightarrow U R_2 w_2 = U R_1 w_1 \\ &\Leftrightarrow \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix} w_2 = \begin{bmatrix} R_1^1 \\ R_1^2 \end{bmatrix} w_1 \\ &\Leftrightarrow R_1^2 w_1 = 0 \text{ and } w_2 = R_1^1 w_1, \end{aligned}$$

with UR_1 conformably partitioned as

$$U\left(\xi, \xi^{-1}\right) R_1\left(\xi, \xi^{-1}\right) = \begin{bmatrix} R_1^1\left(\xi, \xi^{-1}\right) \\ R_1^2\left(\xi, \xi^{-1}\right) \end{bmatrix}.$$

Let

$$R_1^1\left(\xi, \xi^{-1}\right) = {R_1^1}^{-M} \xi^{-M} + \cdots + {R_1^1}^0 + \cdots + {R_1^1}^N \xi^N,$$

with $N, M \in \mathbb{Z}_+$. Applying σ^M to both sides of the equality $w_2 = R_1^1 w_1$, we obtain

$$\left(\sigma^M w_2\right)(k) = \left(\widetilde{R}_1^1(\sigma)\, w_1\right)(k), \quad k \in \mathbb{Z},$$

allowing us to conclude that

$$\left\{ w_1\Big|_{[k_0, +\infty)} = 0 \right\} \Rightarrow \left\{ w_2\Big|_{[k_0+M, +\infty)} = 0 \right\},$$

i.e., w_2 is M-reconstructible, and hence reconstructible, from w_1.

Suppose now that (2.6) does not hold. Then, there exists a trajectory $w_2^* \in \ker R_2\left(\sigma, \sigma^{-1}\right)$, which is non-zero [52]. This trajectory is such that

$$w^* = \left(w_1^* \equiv 0, w_2^*\right) \in \mathfrak{B}.$$

If w_2 were reconstructible from w_1, this would imply that

$$\left. w_2^* \right|_{[k^*,+\infty)} \equiv 0, \quad \forall k^* \in \mathbb{Z},$$

and, consequently, w_2^* would be null in the whole time-axis $\mathbb{Z}$, which is a contradiction. Therefore, if the rank condition (2.6) does not hold, w_2 is not reconstructible from w_1, or, in other words, the reconstructibility of w_2 from w_1 implies that (2.6) holds;

ii) Suppose now that there exist $\widetilde{R}_2$ and $\widetilde{R}_1$ such that $\mathfrak{B}$ is described by

$$\left(\widetilde{R}_2\left(\sigma\right) w_2\right)\left(k\right) = \left(\widetilde{R}_1\left(\sigma\right) w_1\right)\left(k\right), \quad k \in \mathbb{Z},$$

with $\widetilde{R}_2\left(\xi\right)$ satisfying (2.7). Then, there exists an unimodular matrix (over $\mathbb{R}\left[\xi\right]$) $U\left(\xi\right)$ such that [52],

$$U\left(\xi\right) \widetilde{R}_2\left(\xi\right) = \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} \widetilde{R}_2 w_2 = \widetilde{R}_1 w_1 &\Leftrightarrow U\widetilde{R}_2 w_2 = U\widetilde{R}_1 w_1 \\ &\Leftrightarrow \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix} w_2 = \begin{bmatrix} \widetilde{R}_1^1 \\ \widetilde{R}_1^2 \end{bmatrix} w_1 \\ &\Leftrightarrow \widetilde{R}_1^2 w_1 = 0 \text{ and } w_2 = \widetilde{R}_1^1 w_1, \end{aligned}$$

with $U\widetilde{R}_1$ conformably partitioned as

$$U\left(\xi\right) \widetilde{R}_1\left(\xi\right) = \begin{bmatrix} \widetilde{R}_1^1\left(\xi\right) \\ \widetilde{R}_1^2\left(\xi\right) \end{bmatrix}.$$

Thus, if $w_1\left(k\right) = 0$ for $k \in \left[k_0, +\infty\right)$, then

$$\left(\widetilde{R}_1^1\left(\sigma\right) w_1\right)\left(k\right) = 0, \text{ for } k \in \left[k_0, +\infty\right)$$

and hence

$$w_2\left(k\right) = 0, \quad \text{for } k \in \left[k_0, +\infty\right),$$

which allows us to conclude that w_2 is forward-observable from w_1.

Assume now that w_2 is forward-observable from w_1 and let

$$\left(\widehat{R}_2(\sigma) w_2\right)(k) = \left(\widehat{R}_1(\sigma) w_1\right)(k), \quad k \in \mathbb{Z},$$

be a representation of $\mathfrak{B}$. Consider a trajectory $(w_1, w_2) \in \mathfrak{B}$ such that $w_1 \equiv 0$. Then, by the forward-observability of $\mathfrak{B}$, this implies that

$$\forall k_0 \in \mathbb{Z},\ w_2(k) = 0, \quad k \geq k_0,$$

or, in other words, $w_2 \equiv 0$. This means that $\ker \widehat{R}_2(\sigma) = \{0\}$, which is equivalent to say, see [61], that

$$\operatorname{rank} \widehat{R}_2(\lambda) = \text{const}, \quad \forall \lambda \in \mathbb{C} \backslash \{0\}.$$

Note that this also follows immediately from the previous item and from noticing that forward-observability implies reconstructibility.

Let now $U(\xi)$ and $V(\xi)$ be unimodular matrices (over $\mathbb{R}[\xi]$) that bring $\widehat{R}_2$ into its Smith form, i.e.,

$$U\widehat{R}_2 V = \left[\begin{array}{ccc} \xi^{\ell_1} & & \\ & \ddots & \\ & & \xi^{\ell_{q_2}} \\ \hline & 0 & \end{array}\right] =: \left[\begin{array}{c} \Xi \\ \hline 0 \end{array}\right].$$

Then,

$$\begin{aligned} \widehat{R}_2 w_2 = \widehat{R}_1 w_1 &\Leftrightarrow U\widehat{R}_2 w_2 = U\widehat{R}_1 w_1 \\ &\Leftrightarrow \left[\begin{array}{c} \Xi \\ \hline 0 \end{array}\right] V^{-1} w_2 = \begin{bmatrix} \widehat{R}_1^1 \\ \widehat{R}_1^2 \end{bmatrix} w_1, \end{aligned}$$

which is equivalent to

$$\widehat{R}_1^2 w_1 = 0 \quad \text{and} \quad w_2 = V \Xi^{-1} \widehat{R}_1^1 w_1,$$

with $U\widehat{R}_1$ conformably partitioned as

$$U(\xi)\widehat{R}_1(\xi) = \begin{bmatrix} \widehat{R}_1^1(\xi) \\ \widehat{R}_1^2(\xi) \end{bmatrix}.$$

Thus, $\mathfrak{B}$ is also described by

$$\left(\widetilde{R}_2(\sigma)\, w_2\right)(k) = \left(\widetilde{R}_1(\sigma)\, w_1\right)(k),$$

with $\widetilde{R}_2$, $\widetilde{R}_1$ defined by

$$\begin{aligned}\widetilde{R}_2(\xi) &:= \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix}\\ \widetilde{R}_1(\xi) &= \begin{bmatrix} \widetilde{R}_1^1(\xi) \\ \widetilde{R}_1^2(\xi) \end{bmatrix} := \begin{bmatrix} V(\xi)\, \Xi^{-1}(\xi)\, \widehat{R}_1^1(\xi) \\ \widehat{R}_1^2(\xi) \end{bmatrix}.\end{aligned}$$

Now, using the forward-observability property, it is clear that the matrix $\widetilde{R}_1^1 := V\Xi^{-1}\widehat{R}_1^1$ cannot have terms in ξ^{-1}. Moreover, $\widetilde{R}_2(\xi)$ has constant column rank over $\mathbb{C}$. This shows that there exists a representation of $\mathfrak{B}$ with the desired properties.

$\square$

The observability condition (2.5) for behaviors over $\mathbb{Z}$ described by

$$R_2\left(\sigma, \sigma^{-1}\right) w_2 = R_1\left(\sigma, \sigma^{-1}\right) w_1$$

is equivalent to the rank condition

$$\operatorname{rank} R_2\left(\lambda, \lambda^{-1}\right) = q_2, \quad \forall \lambda \in \mathbb{C} \backslash \{0\},$$

see [61]. This coincides with the condition (2.6) in Theorem 2.5.4–i), thus leading to the conclusion that Willems-observability is equivalent to our notion of reconstructibility, rather than to forward-observability.

Proposition 2.5.5 *Let $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ be a time-invariant behavior whose system variable w is partitioned as $w = (w_1, w_2)$. Then w_2 is* Willems-observable *from w_1 if and only if it is* reconstructible *from w_1.* $\diamond$

Note that the definition of observability, given in [56], for systems over $\mathbb{Z}_+$ can be regarded as an adaptation of Willems's definition (2.5), since it means that if w_1 is the null trajectory (i.e., is zero over the time-axis $\mathbb{Z}_+$), then the same happens for w_2. However, that notion can also be seen as an adaptation of our definition of forward-observability.

The situation is summarized in the following table

	$\mathbb{T}$	
Property	$\mathbb{Z}$	$\mathbb{Z}_+$
Forward-observability	(C1)	(C1)
Willems-observability	(C2)	(C1)
Reconstructibility	(C3)$\Leftrightarrow$(C2)	(C3)

where the conditions (C1), (C2) and (C3) are as follows:

(C1) $w_1\big|_{\mathbb{Z}_+} = 0 \Rightarrow w_2\big|_{\mathbb{Z}_+} = 0$;

(C2) $w_1\big|_{\mathbb{Z}} = 0 \Rightarrow w_2\big|_{\mathbb{Z}} = 0$;

(C3) $\exists \delta \geq 0$ s.t. $w_1\big|_{\mathbb{Z}_+} = 0 \Rightarrow w_2\big|_{[\delta,+\infty)} = 0$.

§2.5.2 Reconstructibility and forward-observability of time-invariant state space systems

We start by comparing the behavioral definition of reconstructibility, when applied to time-invariant state space systems, with the classical definition given by Definition 1.4.1. For this purpose, consider a behavior $\mathfrak{B}$ consisting of the set of (x, u, y)-trajectories of an n-dimensional linear and time-invariant state space model, with n states, m inputs and p outputs

$$\begin{cases} (\sigma x)(k) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \qquad k \in \mathbb{Z}. \tag{2.8}$$

Note that, due to time-invariance, complete state reconstructibility is equivalent to complete state reconstructibility at time 0. Thus, it follows from Definition 1.4.1 that the state space system is (completely state) reconstructible if there exists $k_0 \geq 0$ such that

$$\left\{ (u, y)\Big|_{[-k_0,0)} \equiv 0 \right\} \Rightarrow \{x(0) = 0\}. \tag{2.9}$$

Now, due to time-invariance, condition (2.9) is equivalent to

$$\left\{ (u,y)\Big|_{[0,k_0)} \equiv 0 \right\} \Rightarrow \{x(k_0) = 0\}.$$

Moreover, once $u \equiv 0$, the condition $x(k_0) = 0$ clearly implies that

$$x\Big|_{[k_0,+\infty)} \equiv 0,$$

and thus

$$\left\{ (u,y)\Big|_{[0,k_0)} \equiv 0 \right\} \Rightarrow \left\{ x\Big|_{[k_0,+\infty)} \equiv 0 \right\}.$$

This allows us to conclude that complete state reconstructibility implies the existence of $k_0 \geq 0$ such that

$$\left\{ (u,y)\Big|_{[0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ x\Big|_{[k_0,+\infty)} \equiv 0 \right\},$$

which precisely coincides with the k_0-reconstructibility of x from (u,y) as defined in the behavioral framework (cf (2.4)).

In order to see that the opposite implication also holds, assume that the state x is reconstructible from (u,y) in the behavioral sense. Then

$$\left\{ (u,y)\Big|_{[0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ x\Big|_{[\delta,+\infty)} \equiv 0 \right\},$$

for some $\delta \geq 0$. Therefore, taking into account the expressions for

$$y(0), \ldots, y(\delta), y(\delta+1), \ldots$$

and

$$x(\delta), x(\delta+1), \ldots,$$

defining

$$\mathfrak{O}_r := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}, \qquad \mathfrak{K}_r := \ker \mathfrak{O}_r,$$

and letting $\mathfrak{O}_\infty$ and $\mathfrak{K}_\infty$ have the obvious meaning, we have that

$$\{\mathfrak{O}_\infty x(0) = 0\} \Rightarrow \left\{ A^{\delta+\wp} x(0) = 0, \ \wp = 0,1,\ldots \right\}.$$

This means that

$$\mathfrak{K}_\infty \subset \ker A^{\delta+\wp}, \quad \text{for all } \wp = 0,1,\ldots\ .$$

In particular, since for a suitable $\wp$, $\alpha := \delta + \wp \geq n$, this implies that

$$\mathfrak{K}_\infty \subset \ker A^\alpha, \quad \text{for all } \alpha \geq n. \tag{2.10}$$

Now, due to the Cayley-Hamilton theorem, it is clear that $\mathfrak{K}_\infty = \mathfrak{K}_n$ and $\ker A^\alpha = \ker A^n$. Thus, (2.10) is equivalent to the condition

$$\mathfrak{K}_n \subset \ker A^n,$$

which is the classical complete state reconstructibility condition (cf Theorem 1.4.3–iii)).

We can conclude, in this way, that a state space system is reconstructible in the classical sense (Def. 1.4.1) if and only if x is reconstructible from (u, y) in the behavioral sense (Def. 2.5.1).

Note that this conclusion can also be obtained by directly comparing the characterizations of classical and behavioral reconstructibility in terms of rank conditions given in Theorems 1.4.3 and 2.5.4. Indeed, rewriting the equations (2.8) as

$$\begin{bmatrix} \sigma I_n - A \\ C \end{bmatrix} x = \begin{bmatrix} B & 0 \\ -D & I_p \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix},$$

clearly, by Theorem 2.5.4, x is reconstructible from (u, y), in the behavioral sense, if and only if

$$\operatorname{rank} \begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \setminus \{0\}.$$

This coincides with the complete state reconstructibility condition ii) in Theorem 1.4.3, which amounts to say that if $\lambda \in \mathbb{C}$ is an unobservable mode of (C, A), then $\lambda = 0$.

Contrary to what happens with reconstructibility, the characterization of forward-observability for state space systems over $\mathbb{Z}$ does not coincide with the complete state observability condition ii) of Theorem 1.4.4,

$$\operatorname{rank} \begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}.$$

This is illustrated in the following example.

Example 2.5.6 *Consider the a state space system with no inputs, state $x = [x_1 \;\; x_2]^T$ and output y, described by*

$$\begin{cases} (\sigma x)(k) = Ax(k) \\ \quad y(k) = Cx(k) \end{cases} \qquad k \in \mathbb{Z},$$

with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

It turns out that the state x is forward-observable from the output y, since the system trajectories satisfy $x_1 = 0$ and $x_2 = y$. However,

$$\begin{bmatrix} \lambda I_2 - A \\ C \end{bmatrix}$$

has a rank drop for $\lambda = 0$. Nevertheless, the description

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \sigma - 1 \\ 0 & 1 \end{bmatrix}}_{\widetilde{R}_2(\sigma)} x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y$$

is such that

$$\operatorname{rank} \widetilde{R}_2(\lambda) = 2, \quad \forall \lambda \in \mathbb{C},$$

it satisfies the condition of Theorem 2.5.4–ii). $\diamond$

Thus, forward-observability cannot be directly characterized in terms of the unobservable modes of (C, A). However, the following result holds.

Proposition 2.5.7 *Consider the behavior $\mathfrak{B}$ (over $\mathbb{Z}$) described by the state space equations* (2.8)*. Then x is forward-observable from (u, y) if and only if there exists a suitable change of variable $\bar{x}(k) = Sx(k)$, where S is an invertible $n \times n$ matrix, such that, in the transformed system*

$$\begin{cases} \sigma \bar{x}_1 &= A_1 \bar{x}_1 + B_1 u \\ \bar{x}_2 &= 0 \\ y &= C_1 \bar{x}_1 + D u, \end{cases} \qquad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

with

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = SAS^{-1}, \quad \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = SB, \quad \begin{bmatrix} C_1 & 0 \end{bmatrix} = CS^{-1},$$

and A_1, B_1 and C_1 have sizes $n_1 \times n_1$, $n_1 \times m$ and $p \times n_1$, respectively, the pair (C_1, A_1) is observable, that is,

$$\operatorname{rank} \begin{bmatrix} \lambda I_{n_1} - A_1 \\ C_1 \end{bmatrix} = n_1, \quad \forall \lambda \in \mathbb{C}. \tag{2.11}$$

Proof.

($\Leftarrow$): This implication is obvious due to the equivalence between the forward-observability of $\bar{x}$ and x (both from (u, y)), and to the characterization of forward-observability (given in Theorem 2.5.4–ii));

($\Rightarrow$): Assume that x is forward-observable from (u, y). Let S be a change of coordinates such that $\bar{A} := SAS^{-1}$ is in the Jordan form. Partition $\bar{A}$ as

$$\bar{A} = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right],$$

where the block A_1 (of size $n_1 \times n_1$) contains all the non-zero eigenvalues of A and the block A_2 (of size $n_2 \times n_2$) has only null eigenvalues. Partition further, accordingly, the matrices $\bar{C} = CS^{-1}$ and $\bar{B} = SB$ as $\bar{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, and the variable $\bar{x} = Sx$ as $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$. Then the original state space equations are equivalent to:

$$\sigma \bar{x}_1 = A_1 \bar{x}_1 + B_1 u \tag{2.12a}$$

$$\sigma \bar{x}_2 = A_2 \bar{x}_2 + B_2 u \tag{2.12b}$$

$$y = C_1 \bar{x}_1 + C_2 \bar{x}_2 + Du. \tag{2.12c}$$

Without loss of generality, we may assume that A_2 is formed only by one Jordan block (since the reasoning that follows can be carried out independently for all the blocks of A_2). Thus,

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & & \vdots \\ \vdots & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

Let

$$\bar{x}_2 = \begin{bmatrix} x^1 \\ \vdots \\ x^{n_2} \end{bmatrix} \qquad \text{and} \qquad B_2 = \begin{bmatrix} b^1 \\ \vdots \\ b^{n_2} \end{bmatrix},$$

where b^j denotes the j^{th} row of B_2. Then Eq. (2.12b) becomes

$$\left\{\begin{array}{ccc} x^1(k+1) & = & x^2(k)+b^1u(k) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x^{n_2-1}(k+1) & = & x^{n_2}(k)+b^{n_2-1}u(k) \\ x^{n_2}(k+1) & = & b^{n_2}u(k) \end{array}\right. \qquad k\in\mathbb{Z}. \tag{2.13}$$

Note also that (2.12a) may be written as

$$\bar{x}_1(k-1)=A_1^{-1}\bar{x}_1(k)-A_1^{-1}B_1u(k-1), \quad k\in\mathbb{Z}, \tag{2.14}$$

due to the fact that A_1 is non-singular.

Now, consider a trajectory $(\bar{x},u,y)$ such that

$$u(k)=0, \quad y(k)=0, \quad k\geq 0.$$

It follows from Eq. (2.14) that the values of u may be assigned freely in $(-\infty,-1]$, since given a value of $u(k-1)$ and $\bar{x}_1(k)$, a compatible value of $\bar{x}_1(k-1)$ may be computed by (2.14). As for $\bar{x}_2$, its values may be obtained from u. Assume that $b^{n_2}\neq 0\in\mathbb{R}^{1\times m}$. Choose $u(-1)=v\notin\ker b^{n_2}$. Then,

$$x^{n_2}(0)=b^{n_2}v\neq 0.$$

This contradicts the fact that $\bar{x}$, and consequently that $x=S^{-1}\bar{x}$, is forward-observable from (u,y). Thus, b^{n_2} must be zero, and (2.13) may be written as

$$\left\{\begin{array}{ccc} x^1(k+1) & = & x^2(k)+b^1u(k) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ x^{n_2-1}(k+1) & = & b^{n_2-1}u(k) \\ x^{n_2}(k) & = & 0 \end{array}\right. \qquad k\in\mathbb{Z}.$$

Repeating the previous procedure we conclude that

$$\left\{\begin{array}{c} x^1(k)=0 \\ \vdots \\ x^{n_2-1}(k)=0 \end{array}\right. \qquad k\in\mathbb{Z}.$$

Thus, (2.12b) may be replaced by

$$x_2\,(k) = 0, \quad k \in \mathbb{Z}.$$

This means that Eqs. (2.12a–2.12c) have the desired form

$$\begin{cases} \sigma \bar{x}_1 & = & A_1 \bar{x}_1 + B_1 u \\ \bar{x}_2 & = & 0 \\ y & = & C_1 \bar{x}_1 + Du. \end{cases}$$

Note finally that, since forward-observability is a particular case of reconstructibility,

$$\operatorname{rank} \left[\begin{array}{c|c} \lambda I_{n_1} - A_1 & 0 \\ \hline 0 & I_{n_2} \\ \hline C_1 & 0 \end{array} \right] = n, \quad \forall \lambda \in \mathbb{C} \backslash \{0\}\,.$$

Since $\lambda = 0$ is not an eigenvalue of A_1, this implies that

$$\operatorname{rank} \begin{bmatrix} \lambda I_{n_1} - A_1 \\ C_1 \end{bmatrix} = n_1, \quad \forall \lambda \in \mathbb{C}.$$

□

§2.6 Conclusion

This chapter is divided into two main parts. The first one is devoted to presenting the essential background material concerning time-invariant systems, within the behavioral setting. Here, our main source has been the work of Jan C. Willems and co-authors. The second part of this chapter consists of Section 2.5 and contains original material concerning the properties of reconstructibility and forward-observability for systems over $\mathbb{Z}$, which was defined during our research.

Part B

Periodic behaviors

"Imagination is more important than knowledge..."
— Albert Einstein

"Everything you can imagine is real."
— Pablo Picasso

3

Periodic behaviors and their representations

We focus on periodic behaviors, which allow a kernel-type representation, called *P-periodic kernel representation (P-PKR)*. An important tool in the study carried out here is the *lifted behavior* introduced in [47], which is a time-invariant behavior whose trajectories are constructed from the trajectories of the original periodic behavior, similar to what happens in [55] within the classical approach. Based on the relation between the representations of periodic behaviors and the representations of the associated time-invariant behaviors obtained by lifting, we characterize P-periodic kernel representations with respect to equivalence and minimality. Further, we introduce latent variable (and, in particular, image) representations in the periodic context and obtain a latent variable elimination procedure using lifted behaviors.

§3.1 Periodic behaviors - the lifting technique

While the behavior $\mathfrak{B}$ of a time-invariant system over $\mathbb{Z}$ is characterized by its invariance under the time shift (and its inverse), which amounts to $\sigma\mathfrak{B} = \mathfrak{B}$, P-periodic behaviors are required to be invariant only with respect to the P-th power of the shift, and its inverse.

Definition 3.1.1 [47] *A system Σ is said to be P-periodic, with $P \in \mathbb{N}$, if its behavior $\mathfrak{B}$ satisfies*

$$\sigma^P \mathfrak{B} = \mathfrak{B}, \tag{3.1}$$

but not $\sigma^Q \mathfrak{B} = \mathfrak{B}$*, for* $Q \in \mathbb{N}$ *smaller than* P. $\diamond$

Note that (3.1) is equivalent to have

$$\sigma^{\pm P} w \in \mathfrak{B}, \quad \forall w \in \mathfrak{B}.$$

Example 3.1.2 *The behavior* $\mathfrak{B} \subset \mathbb{R}^{\mathbb{Z}}$ *defined by*

$$w(2k) = w(2k-1), \quad k \in \mathbb{Z}, \tag{3.2}$$

describes a 2*-periodic system since*

$$\begin{aligned}\left(\sigma^{\pm 2} w\right)(2k) = w(2k \pm 2) = w(2(k \pm 1)) &\overset{by\ (3.2)}{=} w(2(k \pm 1) - 1) \\ = w(2k - 1 \pm 2) = \left(\sigma^{\pm 2} w\right)(2k-1), &\quad k \in \mathbb{Z}.\end{aligned}$$

Thus $w \in \mathfrak{B} \Rightarrow \sigma^{\pm 2} w \in \mathfrak{B}$ *and hence* $\sigma^2 \mathfrak{B} = \mathfrak{B}$*. However,* $\sigma \mathfrak{B} \neq \mathfrak{B}$ *because, for instance, the trajectory* w *defined by*

$$\begin{cases} w(2k) = 2k \\ w(2k-1) = 2k \end{cases}, \qquad k \in \mathbb{Z}$$

belongs to $\mathfrak{B}$*, but* σw *does not.* $\diamond$

As previously mentioned, a widely common approach when dealing with periodic systems is to relate them with some suitable time-invariant ones as, for instance, the invariant formulation of [55] presented in Chapter 1, or the *lifted* and the *twisted* systems, introduced in [31, 42, 47–50, 58, 62]. Here, following [47], we consider the linear map

$$L_P : (\mathbb{R}^q)^{\mathbb{Z}} \to \left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}},$$

defined by

$$(L_P w)(k) := \begin{bmatrix} w(Pk) \\ \vdots \\ w(Pk + P - 1) \end{bmatrix}, \quad P \in \mathbb{N},$$

and associate with a P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ the *lifted system*, $\Sigma_P^L = \left(\mathbb{Z}, \mathbb{R}^{Pq}, L_P \mathfrak{B}\right)$, with behavior, from here on referred as *lifted behavior*,

$$L_P \mathfrak{B} = L_P(\mathfrak{B}) := \left\{ \widetilde{w} \in \left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}} \mid \widetilde{w} = L_P w,\ w \in \mathfrak{B} \right\}.$$

Note that in this definition of the *lifted trajectory* $L_P w$, the first component is $w(Pk)$ rather than $w(Pk+1)$ as adopted in [42, 47].

As stated in [47], the lifting map L_P satisfies the properties presented in the next propositions.

Proposition 3.1.3

i) $L_P \sigma^P = \sigma L_P$;

ii) *L_P is a homeomorphism; consequently L_P is a closed map.*

$\diamond$

This allows to relate a P-periodic system with the corresponding lifted system, yielding the following result.

Proposition 3.1.4

i) *Σ is P-periodic if and only if P is the smallest positive integer for which Σ_P^L is time-invariant;*

ii) *$\mathfrak{B}$ is linear if and only if $L_P \mathfrak{B}$ is linear;*

iii) *$\mathfrak{B}$ is closed if and only if $L_P \mathfrak{B}$ is closed.*[1]

$\diamond$

In the sequel, for simplicity of notation, we shall drop the subscript P in L_P and Σ_P^L.

§3.2 Kernel-type representations

Following the behavioral spirit, the Definition 3.1.1 of P-periodic system is not given in terms of equations representing the system. It has been shown in [47] that $\mathfrak{B}$ is a σ^P-invariant linear closed subspace of $\left(\mathbb{R}^q\right)^{\mathbb{Z}}$ if and only if

$$\mathfrak{B} \sim \left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = 0, \quad t = 0, \ldots, P-1, \ k \in \mathbb{Z}, \tag{3.3}$$

where each $R_t\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g_t \times q}\left[\xi, \xi^{-1}\right]$ is a Laurent-polynomial matrix in the indeterminate ξ. Notice that the Laurent-polynomial matrices R_t need not have the same

[1]Recall that a subspace of $\left(\mathbb{R}^r\right)^{\mathbb{Z}}$ is closed if it is closed in the topology of pointwise convergence.

number of rows (in fact we could even have some g_t equal to zero, meaning that the corresponding matrix R_t would be void and no restrictions were imposed at the time instants $Pk+t$).

Example 3.2.1 *Consider the 2-periodic system Σ with behavior $\mathfrak{B} \subset (\mathbb{R}^2)^{\mathbb{Z}}$ defined by*

$$\mathfrak{B} \sim \left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(2k+t)=0, \quad t=0,1,\ k\in\mathbb{Z},$$

with

$$R_0\left(\xi,\xi^{-1}\right)=\begin{bmatrix} \xi^2-\xi & \xi^3 \end{bmatrix} \in \mathbb{R}^{1\times 2}\left[\xi,\xi^{-1}\right],$$

$$R_1\left(\xi,\xi^{-1}\right)=\begin{bmatrix} 1-\xi & \xi^3-\xi \\ 2\xi^2 & \xi-\xi^2 \end{bmatrix} \in \mathbb{R}^{2\times 2}\left[\xi,\xi^{-1}\right].$$

This definition leads to the periodically time-varying difference equations

$$\left(\begin{bmatrix} \sigma^2-\sigma & \sigma^3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)(2k)=0$$

$$\left(\begin{bmatrix} 1-\sigma & \sigma^3-\sigma \\ 2\sigma^2 & \sigma-\sigma^2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)(2k+1)=0,$$

that is,

$$\begin{aligned} w_1(2k+2)-w_1(2k+1)+w_2(2k+3)&=0 \\ w_1(2k+1)-w_1(2k+2)+w_2(2k+4)-w_2(2k+2)&=0 \\ 2w_1(2k+3)+w_2(2k+2)-w_2(2k+3)&=0. \end{aligned}$$

◇

Observe now that, since

$$\left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk+t)=\left(\left(\sigma^t R_t\left(\sigma,\sigma^{-1}\right)\right)w\right)(Pk),$$

Eq. (3.3) can also be written as

$$\left(R\left(\sigma,\sigma^{-1}\right)w\right)(Pk)=0, \quad k\in\mathbb{Z}, \tag{3.4}$$

where

$$R\left(\xi,\xi^{-1}\right):=\begin{bmatrix} R_0\left(\xi,\xi^{-1}\right) \\ \xi R_1\left(\xi,\xi^{-1}\right) \\ \vdots \\ \xi^{P-1}R_{P-1}\left(\xi,\xi^{-1}\right) \end{bmatrix} \in \mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right], \tag{3.5}$$

with $g := \sum_{t=0}^{P-1} g_t$. Analogously to the time-invariant case, although with some abuse of language, we refer to (3.4) as a *P-periodic kernel representation (P-PKR)* and to the matrix $R(\xi, \xi^{-1})$ as a *P-PKR matrix* of the corresponding behavior.

Remark 3.2.2 *Note that by considering the P-PKR matrix R, we are ignoring the partition that is initially given by the matrices $R_0, \ldots, R_{P-1}$. Indeed, this partition is irrelevant, as can be seen in* Example 3.2.1. *In this example $P = 2$, R_0 has one row and R_1 has two rows and the final description consists of three difference equations, which could be obtained as well by taking adequately defined R_0 and R_1 matrices with two and one row, respectively.* $\diamond$

Since, for a given P, any integer i has a unique representation $i = j + n_i P$ with $n_i \in \mathbb{Z}$ and $j \in \{0, \ldots, P-1\}$, the following decomposition of $R(\xi, \xi^{-1})$

$$\begin{aligned} R\left(\xi, \xi^{-1}\right) &= \sum_{i \in \Im \subset \mathbb{Z}} C_i \xi^i = \sum_{j=0}^{P-1} \xi^j \sum_{i \in \Im,\ i = j + n_i P} C_i \xi^{n_i P} \\ &= \sum_{j=0}^{P-1} \xi^j \underbrace{\sum_{n_i \in \mathbb{Z}} C_{j+n_i P} \xi^{n_i P}}_{=: R_j^L(\xi^P, \xi^{-P})} = R^L\left(\xi^P, \xi^{-P}\right) \Omega_{P,q}(\xi) \end{aligned} \tag{3.6}$$

with

$$\Omega_{P,q}(\xi) := \left[\begin{array}{cccc} I_q & \xi I_q & \cdots & \xi^{P-1} I_q \end{array} \right]^T \tag{3.7}$$

and

$$\begin{aligned} R^L\left(\xi, \xi^{-1}\right) &= \left[\begin{array}{cccc} R_0^L(\xi, \xi^{-1}) & R_1^L(\xi, \xi^{-1}) & \cdots & R_{P-1}^L(\xi, \xi^{-1}) \end{array} \right] \\ & R_t^L\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times q}\left[\xi, \xi^{-1}\right], \quad t = 0, \ldots, P-1, \end{aligned} \tag{3.8}$$

is unique.[2]

It follows from the decomposition (3.6–3.8) and the definition of the lifted trajectory Lw associated to w, that (3.4) can be written as

$$\left(R^L\left(\sigma, \sigma^{-1}\right)(Lw)\right)(k) = 0, \quad k \in \mathbb{Z}. \tag{3.9}$$

Taking into account that this reasoning can be reversed, we obtain the following result.

[2]see Appendix A for an explicit expression for the matrix R^L.

Lemma 3.2.3 [47] *A P-periodic behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ is given by a P-PKR if and only if the associated lifted behavior $L\mathfrak{B}$ is given by a kernel representation. Moreover if* (3.4) *is a P-PKR of $\mathfrak{B}$, that is,*

$$\mathfrak{B} \sim \left(R\left(\sigma, \sigma^{-1}\right) w\right)(Pk) = 0, \quad k \in \mathbb{Z},$$

then (3.9) *is a kernel representation of $L\mathfrak{B}$, that is,*

$$L\mathfrak{B} \sim \left(R^L\left(\sigma, \sigma^{-1}\right)(Lw)\right)(k) = 0, \quad k \in \mathbb{Z},$$

or simply

$$L\mathfrak{B} = \ker R^L\left(\sigma, \sigma^{-1}\right),$$

where $R^L(\xi, \xi^{-1}) \in \mathbb{R}^{g \times Pq}[\xi, \xi^{-1}]$, $g = \sum_{t=0}^{P-1} g_t$, is defined as in (3.8). $\diamond$

Example 3.2.4 *Refer to* Example 3.2.1. *By definition the matrix $R(\xi, \xi^{-1})$ is given by*

$$\begin{bmatrix} R_0(\xi, \xi^{-1}) \\ \xi R_1(\xi, \xi^{-1}) \end{bmatrix},$$

that is,

$$\begin{bmatrix} \xi^2 - \xi & \xi^3 \\ \xi - \xi^2 & \xi^4 - \xi^2 \\ 2\xi^3 & \xi^2 - \xi^3 \end{bmatrix}.$$

By decomposing R as in (3.6–3.8), *we obtain*

$$\begin{aligned} R\left(\xi, \xi^{-1}\right) &= \begin{bmatrix} \xi^2 & 0 \\ -\xi^2 & \xi^4 - \xi^2 \\ 0 & \xi^2 \end{bmatrix} + \xi \begin{bmatrix} -1 & \xi^2 \\ 1 & 0 \\ 2\xi^2 & -\xi^2 \end{bmatrix} \\ &= \begin{bmatrix} R_0^L(\xi^2, \xi^{-2}) & R_1^L(\xi^2, \xi^{-2}) \end{bmatrix} \Omega_{2,2}(\xi), \end{aligned}$$

where

$$R_0^L\left(\xi^2, \xi^{-2}\right) = \begin{bmatrix} \xi^2 & 0 \\ -\xi^2 & \xi^4 - \xi^2 \\ 0 & \xi^2 \end{bmatrix}, \quad R_1^L\left(\xi^2, \xi^{-2}\right) = \begin{bmatrix} -1 & \xi^2 \\ 1 & 0 \\ 2\xi^2 & -\xi^2 \end{bmatrix}.$$

Therefore the matrix R^L is given by

$$R^L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} R_0^L(\xi, \xi^{-1}) & R_1^L(\xi, \xi^{-1}) \end{bmatrix} = \left[\begin{array}{cc|cc} \xi & 0 & -1 & \xi \\ -\xi & \xi^2 - \xi & 1 & 0 \\ 0 & \xi & 2\xi & -\xi \end{array}\right].$$

$\diamond$

Recalling the established relation between a P-periodic behavior and its associated lifted behavior, and invoking Theorem 2.2.3, we can immediately conclude that for P-periodic behaviors $\mathfrak{B}$ and $\mathfrak{B}'$,

$$\mathfrak{B} \subseteq \mathfrak{B}'$$

if and only if any matrices $R^L(\xi, \xi^{-1})$ and $R'^L(\xi, \xi^{-1})$, that represent the corresponding lifted behaviors $L\mathfrak{B}$ and $L\mathfrak{B}'$, respectively, are related by

$$R'^L\left(\xi, \xi^{-1}\right) = V\left(\xi, \xi^{-1}\right) R^L\left(\xi, \xi^{-1}\right),$$

for some Laurent-polynomial matrix $V(\xi, \xi^{-1})$. This constitutes an indirect characterization of behavior inclusion for the periodic case. However, our next result provides a more direct condition, since it is stated in terms of the P-PKR matrices themselves.

Theorem 3.2.5 *Let $\mathfrak{B}, \mathfrak{B}' \subset (\mathbb{R}^q)^{\mathbb{Z}}$ be two P-periodic behaviors given by the P-PKR matrices $R(\xi, \xi^{-1})$ and $R'(\xi, \xi^{-1})$, respectively. Then $\mathfrak{B} \subseteq \mathfrak{B}'$ if and only if there exists a Laurent-polynomial matrix $V(\xi, \xi^{-1})$ such that*

$$R'\left(\xi, \xi^{-1}\right) = V\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right). \tag{3.10}$$

Proof. Assume that $\mathfrak{B} \subseteq \mathfrak{B}'$. By Theorem 2.2.3, the matrices $R^L(\xi, \xi^{-1})$ and $R'^L(\xi, \xi^{-1})$, that represent the corresponding lifted behaviors, are related by

$$R'^L\left(\xi, \xi^{-1}\right) = V\left(\xi, \xi^{-1}\right) R^L\left(\xi, \xi^{-1}\right), \tag{3.11}$$

for some Laurent-polynomial matrix $V(\xi, \xi^{-1})$. Note now that (3.11) may also be written as

$$R'^L\left(\xi^P, \xi^{-P}\right) = V\left(\xi^P, \xi^{-P}\right) R^L\left(\xi^P, \xi^{-P}\right),$$

immediately yielding (3.10) after right multiplication by $\Omega_{P,q}(\xi)$.

Assume now that (3.10) holds. Taking into account that the previous reasoning can be entirely reversed, it follows that

$$L\mathfrak{B} = \ker R^L\left(\xi, \xi^{-1}\right) \subseteq L\mathfrak{B}' = \ker R'^L\left(\xi, \xi^{-1}\right),$$

which is equivalent of saying that the corresponding inclusion also holds for the associated P-periodic behaviors, i.e., $\mathfrak{B} \subseteq \mathfrak{B}'$. $\square$

As an immediate consequence of Theorem 3.2.5 stands the following theorem.

Theorem 3.2.6 *Let $\mathfrak{B}, \mathfrak{B}' \subset (\mathbb{R}^q)^{\mathbb{Z}}$ be two P-periodic behaviors given by the P-PKR matrices $R(\xi, \xi^{-1})$ and $R'(\xi, \xi^{-1})$, respectively. Then $\mathfrak{B} = \mathfrak{B}'$ if and only if there exist two Laurent-polynomial matrices, $V(\xi, \xi^{-1})$ and $V'(\xi, \xi^{-1})$, such that*

$$R'\left(\xi, \xi^{-1}\right) = V\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right)$$

and

$$R\left(\xi, \xi^{-1}\right) = V'\left(\xi^P, \xi^{-P}\right) R'\left(\xi, \xi^{-1}\right).$$

$\diamond$

Note that, in case the representation matrices R^L and R'^L of the corresponding lifted systems are not full row rank, the matrices V and V' are not unique, as shown in the next example.

Example 3.2.7 *Consider the 2-periodic systems Σ and Σ' with behaviors*

$$\mathfrak{B} := \left\{w | \left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(2k+t) = 0,\ t = 0, 1,\ k \in \mathbb{Z}\right\} \subset \left(\mathbb{R}^2\right)^{\mathbb{Z}},$$

with

$$R_0\left(\xi, \xi^{-1}\right) = \begin{bmatrix} 1-\xi^{-1} & \xi \\ 2\xi & 1-\xi \end{bmatrix} \in \mathbb{R}^{2\times 2}\left[\xi, \xi^{-1}\right],$$

$$R_1\left(\xi, \xi^{-1}\right) = \begin{bmatrix} 1-\xi & \xi^3-\xi \\ 2 & \xi^{-1}-1 \end{bmatrix} \in \mathbb{R}^{2\times 2}\left[\xi, \xi^{-1}\right]$$

and

$$\mathfrak{B}' := \left\{w | \left(R'_t\left(\sigma, \sigma^{-1}\right) w\right)(2k+t) = 0,\ t = 0, 1,\ k \in \mathbb{Z}\right\} \subset \left(\mathbb{R}^2\right)^{\mathbb{Z}},$$

with

$$R'_0\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi^{-3}-\xi^{-2} & -\xi^{-1} \\ 2\xi & 1-\xi \end{bmatrix} \in \mathbb{R}^{2\times 2}\left[\xi, \xi^{-1}\right],$$

$$R'_1\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi-1 & \xi^2 \\ 1-\xi & \xi^3-\xi \end{bmatrix} \in \mathbb{R}^{2\times 2}\left[\xi, \xi^{-1}\right].$$

For system Σ we have

$$R\left(\xi,\xi^{-1}\right)=\begin{bmatrix} R_0 \\ \xi R_1 \end{bmatrix}$$

$$=\begin{bmatrix} 1-\xi^{-1} & \xi \\ 2\xi & 1-\xi \\ \xi-\xi^2 & \xi^4-\xi^2 \\ 2\xi & 1-\xi \end{bmatrix}=\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\xi^2 & \xi^4-\xi^2 \\ 0 & 1 \end{bmatrix}}_{R_0^L(\xi^2,\xi^{-2})}+\xi\underbrace{\begin{bmatrix} -\xi^{-2} & 1 \\ 2 & -1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}}_{R_1^L(\xi^2,\xi^{-2})}$$

and for its associated lifted system, Σ^L,

$$R^L\left(\xi,\xi^{-1}\right)=\begin{bmatrix} R_0^L\left(\xi,\xi^{-1}\right) & R_1^L\left(\xi,\xi^{-1}\right) \end{bmatrix}=\left[\begin{array}{cc|cc} 1 & 0 & -\xi^{-1} & 1 \\ 0 & 1 & 2 & -1 \\ -\xi & \xi^2-\xi & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right].$$

Analogously we have for Σ' and Σ'^L,

$$R'\left(\xi,\xi^{-1}\right)=\begin{bmatrix} \xi^{-3}-\xi^{-2} & -\xi^{-1} \\ 2\xi & 1-\xi \\ \xi^2-\xi & \xi^3 \\ \xi-\xi^2 & \xi^4-\xi^2 \end{bmatrix}=\begin{bmatrix} -\xi^{-2} & 0 \\ 0 & 1 \\ \xi^2 & 0 \\ -\xi^2 & \xi^4-\xi^2 \end{bmatrix}+\xi\begin{bmatrix} \xi^{-4} & -\xi^{-2} \\ 2 & -1 \\ -1 & \xi^2 \\ 1 & 0 \end{bmatrix}$$

and

$$R'^L\left(\xi,\xi^{-1}\right)=\left[\begin{array}{cc|cc} -\xi^{-1} & 0 & \xi^{-2} & -\xi^{-1} \\ 0 & 1 & 2 & -1 \\ \xi & 0 & -1 & \xi \\ -\xi & \xi^2-\xi & 1 & 0 \end{array}\right].$$

Therefore it is possible to conclude that $\mathfrak{B}\subseteq\mathfrak{B}'$ and $\mathfrak{B}'\subseteq\mathfrak{B}$, i.e., $\mathfrak{B}=\mathfrak{B}'$, since

$$R'^L=VR^L \quad \text{and} \quad R^L=V'R'^L,$$

where V and V' are given by

$$V\left(\xi,\xi^{-1}\right)=\begin{bmatrix} -\xi^{-1} & -\alpha_1 & 0 & \alpha_1 \\ 0 & 1-\alpha_2 & 0 & \alpha_2 \\ \xi & -\alpha_3 & 0 & \alpha_3 \\ 0 & -\alpha_4 & 1 & \alpha_4 \end{bmatrix} \tag{3.12}$$

and

$$V'\left(\xi,\xi^{-1}\right)=\begin{bmatrix}\beta_1\xi^2-\xi & 0 & \beta_1 & 0\\ \beta_2\xi^2 & 1 & \beta_2 & 0\\ \beta_3\xi^2 & 0 & \beta_3 & 1\\ \beta_4\xi^2 & 1 & \beta_4 & 0\end{bmatrix},$$

for any real numbers α_i *and* β_j, $i,j=1,\ldots,4$, *and are hence not unique.* ◇

In case R and R' have the same number of rows, it is possible to prove that V and V' can be taken to be unimodular. This yields the following fundamental result, which is the counterpart for P-periodic behaviors of a similar result for the time-invariant case, [52, Theorem 3.6.2].

Theorem 3.2.8 *Let* $\mathfrak{B},\mathfrak{B}'\subset\left(\mathbb{R}^q\right)^{\mathbb{Z}}$ *be two* P*-periodic behaviors given by the* P*-PKR matrices* $R\left(\xi,\xi^{-1}\right)$ *and* $R'\left(\xi,\xi^{-1}\right)$*, respectively, possessing the same number,* g*, of rows. Then* $\mathfrak{B}=\mathfrak{B}'$ *if and only if there exists a unimodular matrix* $U\left(\xi,\xi^{-1}\right)$ *such that*

$$R'\left(\xi,\xi^{-1}\right)=U\left(\xi^P,\xi^{-P}\right)R\left(\xi,\xi^{-1}\right). \tag{3.13}$$

Proof. By [52, Theorem 3.6.2], the two corresponding lifted behaviors, $L\mathfrak{B}$ and $L\mathfrak{B}'$, are the same (and therefore the same happens with the P-periodic behaviors $\mathfrak{B}$ and $\mathfrak{B}'$) if and only if there exists a unimodular matrix $U\left(\xi,\xi^{-1}\right)\in\mathbb{R}^{g\times g}\left[\xi,\xi^{-1}\right]$ such that $R'^L\left(\xi,\xi^{-1}\right)=U\left(\xi,\xi^{-1}\right)R^L\left(\xi,\xi^{-1}\right)$. By changing the indeterminate from ξ to ξ^P this is equivalent to

$$R'^L\left(\xi^P,\xi^{-P}\right)=U\left(\xi^P,\xi^{-P}\right)R^L\left(\xi^P,\xi^{-P}\right).$$

Finally, due to the uniqueness of decomposition (3.6–3.8), this is also equivalent to

$$R'^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}\left(\xi\right)=U\left(\xi^P,\xi^{-P}\right)R^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}\left(\xi\right),$$

yielding (3.13). □

Example 3.2.9 *Recall* Example 3.2.7. *Taking in* (3.12), $\alpha_1=1$ *and* $\alpha_2=\alpha_3=\alpha_4=0$, *we obtain a unimodular matrix*

$$U\left(\xi,\xi^{-1}\right)=\begin{bmatrix}-\xi^{-1} & -1 & 0 & 1\\ 0 & 1 & 0 & 0\\ \xi & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},$$

such that (3.13) *holds.* ◇

Consider now the issue of minimality of representations within the periodic case.

Definition 3.2.10 *A P-PKR (matrix) $R \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$ of a P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is said to be* minimal *if for any other representation $R' \in \mathbb{R}^{g'\times q}[\xi,\xi^{-1}]$ of Σ, there holds $g \le g'$.* $\diamond$

This definition is analogous to the one given for the time-invariant case, according to which a representation is minimal if it has a minimum number of equations. However, contrary to what happens for the time-invariant case, see [52], a minimal P-PKR matrix may not have full row rank.

It is easy to check that a P-PKR, $R(\xi,\xi^{-1})$, of a P-periodic system Σ is minimal if and only if the same happens for the corresponding representation $R^L(\xi,\xi^{-1})$ of the associated (time-invariant) lifted system Σ^L. Thus $R(\xi,\xi^{-1})$ is minimal if and only if $R^L(\xi,\xi^{-1})$ is of full row rank over $\mathbb{R}[\xi,\xi^{-1}]$. The next lemma translates this property in terms of the matrix $R(\xi,\xi^{-1})$ itself.

Lemma 3.2.11 *Let $P \in \mathbb{N}$ and $R(\xi,\xi^{-1}) \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$. Consider the corresponding matrix $R^L(\xi,\xi^{-1}) \in \mathbb{R}^{g\times Pq}[\xi,\xi^{-1}]$ given by (3.6–3.8). Then, the following conditions are equivalent:*

i) *$R^L(\xi,\xi^{-1})$ has full row rank over $\mathbb{R}[\xi,\xi^{-1}]$;*

ii) *$R(\xi,\xi^{-1})$ has full row rank over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$.*[3]

Proof. Recall that the decomposition (3.6–3.8) of matrices $R(\xi,\xi^{-1})$, over $\mathbb{R}[\xi,\xi^{-1}]$, as a product

$$R\left(\xi,\xi^{-1}\right) = R^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}(\xi), \tag{3.14}$$

is unique. From here we can immediately conclude that ii) $\Rightarrow$ i). In order to see that i) $\Rightarrow$ ii), assume that ii) does not hold, i.e., that there exists a non-zero row $r\left(\xi^P,\xi^{-P}\right) \in \mathbb{R}^{1\times g}\left[\xi^P,\xi^{-P}\right]$ such that

$$r\left(\xi^P,\xi^{-P}\right) R\left(\xi,\xi^{-1}\right) = 0.$$

Then, pre-multiplying both sides of (3.14) by $r\left(\xi^P,\xi^{-P}\right)$ yields that

$$r\left(\xi^P,\xi^{-P}\right) R^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}(\xi) = 0 \in \mathbb{R}^{1\times q}\left[\xi,\xi^{-1}\right],$$

[3] i.e., if $r\left(\xi^P,\xi^{-P}\right) \in \mathbb{R}^{1\times g}\left[\xi^P,\xi^{-P}\right]$ is such that $r\left(\xi^P,\xi^{-P}\right) R\left(\xi,\xi^{-1}\right) = 0$, then $r\left(\xi^P,\xi^{-P}\right) = 0$.

which, due to the uniqueness of decomposition (3.6–3.8), implies that

$$r\left(\xi^P, \xi^{-P}\right) R^L\left(\xi^P, \xi^{-P}\right) = 0 \in \mathbb{R}^{1\times Pq}\left[\xi^P, \xi^{-P}\right],$$

which in turn leads to

$$r\left(\xi, \xi^{-1}\right) R^L\left(\xi, \xi^{-1}\right) = 0 \in \mathbb{R}^{1\times Pq}\left[\xi, \xi^{-1}\right],$$

thus showing that $R^L(\xi, \xi^{-1})$ has not full row rank over $\mathbb{R}[\xi, \xi^{-1}]$. Therefore i) $\Rightarrow$ ii).

$\square$

This result, together with the previous considerations, yields the following characterization of minimality.

Theorem 3.2.12 *Let $R(\xi, \xi^{-1}) \in \mathbb{R}^{g\times q}[\xi, \xi^{-1}]$ be the P-PKR matrix of a P-periodic system Σ. Then $R(\xi, \xi^{-1})$ is a minimal P-PKR if and only if it has full row rank over $\mathbb{R}\left[\xi^P, \xi^{-P}\right]$.* $\diamond$

Example 3.2.13 *The representation $R(\xi, \xi^{-1}) \in \mathbb{R}^{3\times 2}[\xi, \xi^{-1}]$ of* Example 3.2.4 *is minimal. Indeed, although it is clearly not full row rank over $\mathbb{R}[\xi, \xi^{-1}]$, it can be shown that it has full row rank over $\mathbb{R}[\xi^2, \xi^{-2}]$.* $\diamond$

§3.3 Latent variable and image-type representations

Similar to what happens in the time-invariant case, it may be sometimes useful to use latent variables in the description of periodic behaviors.

Definition 3.3.1 *A P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is said to have a latent variable representation if its behavior consists of all the trajectories w that satisfy, together with some latent variable trajectory $v \in \left(\mathbb{R}^\ell\right)^{\mathbb{Z}}$, the following set of equations:*

$$\left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = \left(M_t\left(\sigma, \sigma^{-1}\right) v\right)(Pk+t), \quad t = 0, \ldots, P-1, \ k \in \mathbb{Z} \quad (3.15)$$

where

$$\begin{cases} R_t(\xi, \xi^{-1}) \in \mathbb{R}^{g_t\times q}[\xi, \xi^{-1}] \\ M_t(\xi, \xi^{-1}) \in \mathbb{R}^{g_t\times \ell}[\xi, \xi^{-1}], \quad t = 0, \ldots, P-1, \end{cases}$$

i.e.,

$$\mathfrak{B} = \left\{w \in (\mathbb{R}^q)^{\mathbb{Z}} : \exists v \in \left(\mathbb{R}^\ell\right)^{\mathbb{Z}} \text{ s.t. (3.15) holds}\right\}.$$

$\diamond$

Remark 3.3.2 *Note that if* $\mathfrak{B}$ *has a latent variable representation* (3.15)*, then* $\mathfrak{B}$ *is a* P*-periodic behavior, since*

$$w \in \mathfrak{B} \Rightarrow \left(\sigma^{\pm P} w\right) \in \mathfrak{B},$$

due to the fact that, for each $t = 0, \ldots, P-1$,

$$\left(\sigma^{\pm P} R_t\left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = \left(\sigma^{\pm P} M_t\left(\sigma, \sigma^{-1}\right) v\right)(Pk+t), \ k \in \mathbb{Z},$$

i.e.,

$$\left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(P(k \pm 1)+t) = \left(M_t\left(\sigma, \sigma^{-1}\right) v\right)(P(k \pm 1)+t), \quad k \in \mathbb{Z}.$$

◇

Example 3.3.3 *Consider, as already done in* Section 1.1*, a system* Σ *described by the discrete* P*-periodic state space model*

$$\left\{\begin{array}{l} (\sigma x)(k) = A(k)\,x(k) + B(k)\,u(k) \\ \\ \quad y(k) = C(k)\,x(k) + D(k)\,u(k) \end{array}\right. \qquad k \in \mathbb{Z}, \tag{3.16}$$

with, for $P \in \mathbb{N}$,

$$\begin{array}{ll} A(k+P) = A(k) & \qquad C(k+P) = C(k) \\ B(k+P) = B(k) & \qquad D(k+P) = D(k)\,. \end{array}$$

Letting $w := \left[\begin{array}{cc} u^T & y^T \end{array}\right]^T$ *and* $v := x$*, we may rewrite* (3.16) *as*

$$\left[\begin{array}{cc} B(k) & 0 \\ -D(k) & I_p \end{array}\right] w(k) = \left[\begin{array}{c} \sigma I_n - A(k) \\ C(k) \end{array}\right] v(k), \quad k \in \mathbb{Z},$$

or simply

$$\left(R_k\left(\sigma, \sigma^{-1}\right) w\right)(k) = \left(M_k\left(\sigma, \sigma^{-1}\right) v\right)(k), \quad k \in \mathbb{Z}, \tag{3.17}$$

by taking

$$R_k\left(\xi, \xi^{-1}\right) := \left[\begin{array}{cc} B(k) & 0 \\ -D(k) & I_p \end{array}\right], \quad M_k\left(\xi, \xi^{-1}\right) := \left[\begin{array}{c} \xi I_n - A(k) \\ C(k) \end{array}\right].$$

Since the periodicity of matrices $A(\cdot)$*,* $B(\cdot)$*,* $C(\cdot)$ *and* $D(\cdot)$ *leads to*

$$R_{k+P}\left(\xi, \xi^{-1}\right) = R_k\left(\xi, \xi^{-1}\right), \qquad M_{k+P}\left(\xi, \xi^{-1}\right) = M_k\left(\xi, \xi^{-1}\right),$$

we may write down (3.17) *as*

$$\left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = \left(M_t\left(\sigma, \sigma^{-1}\right) v\right)(Pk+t), \quad t = 0, \ldots, P-1, \ k \in \mathbb{Z}.$$

◇

Note that equations (3.15) can be written as

$$\left(R\left(\sigma,\sigma^{-1}\right)w\right)(Pk)=\left(M\left(\sigma,\sigma^{-1}\right)v\right)(Pk), \quad k\in\mathbb{Z}, \tag{3.18}$$

where $R\in\mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right]$, $g:=\sum_{t=0}^{P-1}g_t$, is defined as in (3.5) and M is defined analogously by

$$M\left(\xi,\xi^{-1}\right):=\begin{bmatrix} M_0\left(\xi,\xi^{-1}\right)\\ \xi M_1\left(\xi,\xi^{-1}\right)\\ \vdots\\ \xi^{P-1}M_{P-1}\left(\xi,\xi^{-1}\right)\end{bmatrix}\in\mathbb{R}^{g\times\ell}\left[\xi,\xi^{-1}\right]. \tag{3.19}$$

We denote the latent variable representation (3.18) by (R,M) and, from here on, will refer to it as *P-periodic latent variable representation (P-PLVR).*

By decomposing matrices R and M as

$$\begin{aligned} R\left(\xi,\xi^{-1}\right)&=R^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}(\xi)\\ M\left(\xi,\xi^{-1}\right)&=M^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,\ell}(\xi)\end{aligned} \tag{3.20}$$

and recalling the definition of lifted trajectories, it is straightforward that (3.18) is successively equivalent to

$$\left(R^L\left(\sigma^P,\sigma^{-P}\right)\Omega_{P,q}(\sigma)w\right)(Pk)=\left(M^L\left(\sigma^P,\sigma^{-P}\right)\Omega_{P,\ell}(\sigma)v\right)(Pk), \quad k\in\mathbb{Z}$$

and

$$R^L\left(\sigma^P,\sigma^{-P}\right)\begin{bmatrix} w(Pk)\\ \vdots\\ w(Pk+P-1)\end{bmatrix}=M^L\left(\sigma^P,\sigma^{-P}\right)\begin{bmatrix} v(Pk)\\ \vdots\\ v(Pk+P-1)\end{bmatrix}, \quad k\in\mathbb{Z},$$

and therefore to

$$\left(R^L\left(\sigma,\sigma^{-1}\right)\widetilde{w}\right)(k)=\left(M^L\left(\sigma,\sigma^{-1}\right)\widetilde{v}\right)(k), \quad k\in\mathbb{Z}, \tag{3.21}$$

where $\widetilde{w}=Lw$ and $\widetilde{v}=Lv$.

This reasoning allows us to obtain the following result.

Lemma 3.3.4 *A P-periodic behavior $\mathfrak{B}\subset\left(\mathbb{R}^q\right)^{\mathbb{Z}}$ is given by a P-PLVR if and only if the associated lifted behavior $L\mathfrak{B}$ is given by a latent variable representation (LVR). Moreover, if (R,M) is a P-PLVR of $\mathfrak{B}$, then $\left(R^L,M^L\right)$ is a LVR of $L\mathfrak{B}$.* $\diamond$

Remark 3.3.5 *Note that not every LVR, $\widetilde{R}\widetilde{w} = \widetilde{M}\widetilde{v}$, for the time-invariant behavior $L\mathfrak{B}$ corresponds to a P-PLVR for $\mathfrak{B}$, since the number of components of the latent variable $\widetilde{v}$ must be a multiple of P. Nevertheless, if a LVR $(\widetilde{R}, \widetilde{M})$ of $L\mathfrak{B}$ is given, one can always introduce "extra" components in $\widetilde{v}$ (and accordingly zero columns in $\widetilde{M}$) in order to achieve the aforementioned requirement on the cardinality of the components of $\widetilde{v}$.* $\diamond$

An important issue is the elimination of latent variables in order to obtain a description only in terms of the system variables, which was solved in the time-invariant case.

Theorem 3.3.6 (latent variable elimination) *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system. Then $\mathfrak{B}$ has a P-PLVR (R, M) if and only if it has a P-PKR R^*. Given (R, M), R^* may be obtained as*

$$R^*\left(\xi, \xi^{-1}\right) := H\left(\xi^P, \xi^{-P}\right) R\left(\xi, \xi^{-1}\right), \tag{3.22}$$

where $H\left(\xi, \xi^{-1}\right)$ is a minimal left annihilator *of the matrix $M^L\left(\xi, \xi^{-1}\right)$, in the decomposition* (3.20).

Proof.

($\Rightarrow$): Using the relationship with the lifted system (Lemma 3.3.4), and taking into account the results on latent variable elimination for time-invariant systems (see [52]), we obtain the following equivalences, with $k \in \mathbb{Z}$,

$$\begin{aligned} & \mathfrak{B} \sim \left(R\left(\sigma, \sigma^{-1}\right) w\right)(Pk) = \left(M\left(\sigma, \sigma^{-1}\right) v\right)(Pk) \\ \Leftrightarrow\ & L\mathfrak{B} \sim \left(R^L\left(\sigma, \sigma^{-1}\right) \widetilde{w}\right)(k) = \left(M^L\left(\sigma, \sigma^{-1}\right) \widetilde{v}\right)(k) \\ \Leftrightarrow\ & L\mathfrak{B} \sim \left(H\left(\sigma, \sigma^{-1}\right) R^L\left(\sigma, \sigma^{-1}\right) \widetilde{w}\right)(k) = 0 \\ \Leftrightarrow\ & \mathfrak{B} \sim \left(R^*\left(\sigma, \sigma^{-1}\right) w\right)(Pk) = 0, \end{aligned}$$

where as usual $\widetilde{w} = Lw$, $\widetilde{v} = Lv$, $H\left(\xi, \xi^{-1}\right)$ is a *MLA* of $M^L\left(\xi, \xi^{-1}\right)$ and R^* is given by (3.22);

($\Leftarrow$): This implication is obvious. Since every P-PKR is also a P-PLVR with $M = 0$.

$\square$

Example 3.3.7 *Consider the 2-periodic behavior $\mathfrak{B}$ with the P-PLVR*

$$\left(R\left(\xi,\xi^{-1}\right),M\left(\xi,\xi^{-1}\right)\right)=\left(\begin{bmatrix}\xi\\-1\end{bmatrix},\begin{bmatrix}1\\\xi^2\end{bmatrix}\right),$$

that is,

$$\mathfrak{B}=\left\{w\in(\mathbb{R})^{\mathbb{Z}}:\ \exists v\in(\mathbb{R})^{\mathbb{Z}}\ \text{s.t.}\ \left(\begin{bmatrix}\sigma\\-1\end{bmatrix}w\right)(2k)=\left(\begin{bmatrix}1\\\sigma^2\end{bmatrix}v\right)(2k)\right\}.$$

The matrix $M\left(\xi,\xi^{-1}\right)$ decomposes as

$$M\left(\xi,\xi^{-1}\right)=\begin{bmatrix}1&0\\\xi^2&0\end{bmatrix}\Omega_{2,1}\left(\xi\right).$$

Moreover, $H\left(\xi,\xi^{-1}\right)=\begin{bmatrix}1&-\xi^{-1}\end{bmatrix}$ is a MLA *of matrix $M^L\left(\xi,\xi^{-1}\right)=\begin{bmatrix}1&0\\\xi&0\end{bmatrix}$, and the given behavior $\mathfrak{B}$ is therefore also described by the following P-PKR matrix*

$$H\left(\xi^2,\xi^{-2}\right)R\left(\xi,\xi^{-1}\right)=\begin{bmatrix}1&-\xi^{-2}\end{bmatrix}\begin{bmatrix}\xi\\-1\end{bmatrix}=\xi+\xi^{-2},$$

that is,

$$\mathfrak{B}\ \sim\ \left(\left(\sigma+\sigma^{-2}\right)w\right)(2k)=0,\quad k\in\mathbb{Z}.$$

◇

Combining Theorems 3.2.6 and 3.3.6 it is not difficult to obtain the following criterion for the equivalence of two P-PLVR, in the sense that they describe the same behavior for the system variable w.

Theorem 3.3.8 *Let (R,M) and (R',M') be two P-PLVRs describing the behaviors $\mathfrak{B}$ and $\mathfrak{B}'$, respectively. Then $\mathfrak{B}=\mathfrak{B}'$ if and only if there exist two Laurent-polynomial matrices $V\left(\xi,\xi^{-1}\right)$ and $V'\left(\xi,\xi^{-1}\right)$ such that the following relations hold:*

$$H'\left(\xi^P,\xi^{-P}\right)R'\left(\xi,\xi^{-1}\right)=V\left(\xi^P,\xi^{-P}\right)H\left(\xi^P,\xi^{-P}\right)R\left(\xi,\xi^{-1}\right)$$

and

$$H\left(\xi^P,\xi^{-P}\right)R\left(\xi,\xi^{-1}\right)=V'\left(\xi^P,\xi^{-P}\right)H'\left(\xi^P,\xi^{-P}\right)R'\left(\xi,\xi^{-1}\right),$$

where $H'\left(\xi,\xi^{-1}\right)$ and $H\left(\xi,\xi^{-1}\right)$ are, respectively, MLAs *of $M'^L\left(\xi,\xi^{-1}\right)$ and $M^L\left(\xi,\xi^{-1}\right)$, with M'^L and M^L obtained from M' and M as in* (3.20). ◇

A special case of latent variable representation occurs when $R_t = I_q,\ t = 0, \ldots, P-1$. In this case equations (3.15) become

$$w\left(Pk+t\right) = \left(M_t\left(\sigma, \sigma^{-1}\right) v\right)\left(Pk+t\right), \quad t = 0, \ldots, P-1,\ k \in \mathbb{Z}. \tag{3.23}$$

By analogy to what happens in the time-invariant case, although with abuse of language, we shall call this description an image representation.

Definition 3.3.9 *A P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is said to have an image representation if its behavior can be written as*

$$\begin{aligned}\mathfrak{B} = \Big\{ w \in \left(\mathbb{R}^q\right)^{\mathbb{Z}} : \exists v \in \left(\mathbb{R}^{\ell}\right)^{\mathbb{Z}} \text{ s.t. } & w\left(Pk+t\right) \\ &= \left(M_t\left(\sigma, \sigma^{-1}\right) v\right)\left(Pk+t\right), \quad t = 0, \ldots, P-1 \Big\},\end{aligned} \tag{3.24}$$

where $M_t\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{q \times \ell}\left[\xi, \xi^{-1}\right],\ t = 0, \ldots, P-1.$ $\diamond$

Remark 3.3.10 *Analogously to the latent variable representation case if $\mathfrak{B}$ has an image representation* (3.24)*, then is P-periodic.* $\diamond$

Observe that, since

$$R\left(\xi, \xi^{-1}\right) = \begin{bmatrix} R_0 \\ \xi R_1 \\ \vdots \\ \xi^{P-1} R_{P-1} \end{bmatrix} = \begin{bmatrix} I_q \\ \xi I_q \\ \vdots \\ \xi^{P-1} I_q \end{bmatrix} = \Omega_{P,q}\left(\xi\right),$$

then, equations (3.23) can be written as

$$\left(\Omega_{P,q}\left(\sigma\right) w\right)\left(Pk\right) = \left(M\left(\sigma, \sigma^{-1}\right) v\right)\left(Pk\right), \quad k \in \mathbb{Z}, \tag{3.25}$$

where the matrix $M\left(\xi, \xi^{-1}\right)$ is defined as in (3.19). From here on, we will refer to (3.25) as a *P-periodic image representation (P-PIR)* and to the matrix $M\left(\xi, \xi^{-1}\right)$ as a *P-PIR matrix*, of the corresponding behavior $\mathfrak{B}$.

Remark 3.3.11 *Note that, since*

$$R\left(\xi, \xi^{-1}\right) = \Omega_{P,q}\left(\xi\right),$$

then,

$$R^L\left(\xi^P, \xi^{-P}\right) = I_{Pq}$$

and therefore expression (3.21) *reduces to*

$$\widetilde{w}(k) = \left(M^L\left(\sigma, \sigma^{-1}\right)\widetilde{v}\right)(k), \quad k \in \mathbb{Z}, \tag{3.26}$$

with $\widetilde{w} = Lw$ *and* $\widetilde{v} = Lv$. ◇

The following is a consequence of Lemma 3.3.4.

Lemma 3.3.12 *A P-periodic behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ is given by a P-PIR if and only if the associated lifted behavior $L\mathfrak{B}$ is given by an* image representation (IR)*. Moreover, if M is a P-PIR matrix of $\mathfrak{B}$, then M^L is an IR matrix of $L\mathfrak{B}$, i.e.,*

$$L\mathfrak{B} = \left\{\widetilde{w} \in \left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}} : \exists \widetilde{v} \in \left(\mathbb{R}^{P\ell}\right)^{\mathbb{Z}} \text{ s.t. (3.26) holds}\right\}.$$

◇

Note that, since P-PIR are a particular case of P-PLVR, what was said in Remark 3.3.5, concerning the construction of such representations based on representations for the lifted behavior, still applies.

The next example illustrates how to obtain a P-PIR representation for a periodic behavior, given an image representation of its lifted behavior, and puts into evidence the issue with the cardinality of the latent variables mentioned in Remark 3.3.5.

Example 3.3.13 *Consider the 2-periodic behavior $\mathfrak{B}$ of* Example 3.1.2*, with P-PKR*

$$R\left(\xi, \xi^{-1}\right) = 1 - \xi^{-1}.$$

Since

$$R\left(\xi, \xi^{-1}\right) = 1 - \xi^{-1} = \begin{bmatrix} 1 & -\xi^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ \xi \end{bmatrix},$$

its associated lifted behavior $L\mathfrak{B}$ is described by the kernel representation

$$\left(R^L\left(\sigma, \sigma^{-1}\right) \begin{bmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \end{bmatrix}\right)(k) = 0, \quad k \in \mathbb{Z},$$

where

$$R^L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} 1 & -\xi^{-1} \end{bmatrix}.$$

It is also possible to describe this lifted behavior in terms of an image representation, namely

$$L\mathfrak{B} = \operatorname{im} \begin{bmatrix} 1 \\ \sigma \end{bmatrix}.$$

In order to achieve the decomposition in equation (3.20)*, we consider that*

$$L\mathfrak{B} = \operatorname{im} M^L\left(\sigma, \sigma^{-1}\right),$$

with $M^L\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{2\times 2\ell}\left[\xi, \xi^{-1}\right]$*, given by*

$$M^L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} 1 & 0 \\ \xi & 0 \end{bmatrix}.$$

Therefore the original 2*-periodic behavior has a* P*-PIR matrix* M *given by*

$$M\left(\xi, \xi^{-1}\right) = \begin{bmatrix} 1 & 0 \\ \xi^2 & 0 \end{bmatrix} \Omega_{2,1}(\xi) = \begin{bmatrix} 1 & 0 \\ \xi^2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \xi \end{bmatrix} = \begin{bmatrix} 1 \\ \xi^2 \end{bmatrix},$$

that is, the 2*-periodic behavior* $\mathfrak{B}$ *allows the* P*-PIR*

$$\mathfrak{B} = \left\{ w \in (\mathbb{R})^{\mathbb{Z}} : \exists v \in (\mathbb{R})^{\mathbb{Z}} \text{ s.t. } \begin{bmatrix} w(2k) \\ w(2k+1) \end{bmatrix} = \left(M\left(\sigma, \sigma^{-1}\right) v\right)(2k) \right\}. \qquad \diamond$$

The following result particularizes Theorem 3.3.6 to the P-PIR case.

Theorem 3.3.14 (latent variable elimination) *Let* $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ *be a* P*-periodic system. Then* $\mathfrak{B}$ *has a* P*-PIR* M *only if it has a* P*-PKR* $\widehat{R}$*. Given* M*,* $\widehat{R}$ *can be obtained as*

$$\widehat{R}\left(\xi, \xi^{-1}\right) := H\left(\xi^P, \xi^{-P}\right) \Omega_{P,q}(\xi),$$

where H *is a* MLA *of* M^L*.*

Proof. This proof, via lifted system, is a particular case of the proof of the first part of Theorem 3.3.6. Indeed note that we obtain the desired result by letting

$$R\left(\xi, \xi^{-1}\right) = \Omega_{P,q}(\xi)$$

or, equivalently,

$$R^L\left(\xi^P, \xi^{-P}\right) = I_{Pq}.$$

$\square$

Remark 3.3.15 *As the next example shows,*

$$\mathfrak{B} \text{ has } P\text{-PKR} \nRightarrow \mathfrak{B} \text{ has } P\text{-PIR}.$$

$\diamond$

Example 3.3.16 *Consider the* 2*-periodic behavior* $\mathfrak{B}$ *associated with the* P*-PKR*

$$R\left(\xi, \xi^{-1}\right) = \xi^2 - 1,$$

i.e., described by

$$w\left(2k+2\right) = w\left(2k\right), \quad k \in \mathbb{Z}.$$

The corresponding lifted behavior is $L\mathfrak{B} = \ker R^L\left(\sigma, \sigma^{-1}\right)$*, with*

$$R^L\left(\xi, \xi^{-1}\right) = \left[\begin{array}{cc} \xi - 1 & 0 \end{array}\right].$$

This behavior does not allow an image representation. Indeed if this were the case, say, if $L\mathfrak{B} = \operatorname{im} M^L\left(\sigma, \sigma^{-1}\right)$*, then* $R^L\left(\xi, \xi^{-1}\right)$ *should be a* MLA *of* $M^L\left(\xi, \xi^{-1}\right)$*. But in this case also* $\left[\begin{array}{cc} 1 & 0 \end{array}\right]$ *would be an annihilator of* $M^L\left(\xi, \xi^{-1}\right)$*, which is absurd since it cannot be obtained as a multiple of* $R^L\left(\xi, \xi^{-1}\right) = \left[\begin{array}{cc} \xi - 1 & 0 \end{array}\right]$*. Consequently,* $\mathfrak{B}$ *cannot have a* P*-PIR.* $\diamond$

Taking Remark 3.3.11 and Theorem 3.3.8 into account we can conclude that two P-PIR M and M' describe the same behavior if and only if there exist two Laurent-polynomial matrices $V\left(\xi, \xi^{-1}\right)$ and $V'\left(\xi, \xi^{-1}\right)$ such that the following relations hold:

$$H'\left(\xi^P, \xi^{-P}\right) \Omega_{P,q}\left(\xi\right) = V\left(\xi^P, \xi^{-P}\right) H\left(\xi^P, \xi^{-P}\right) \Omega_{P,q}\left(\xi\right)$$

and

$$H\left(\xi^P, \xi^{-P}\right) \Omega_{P,q}\left(\xi\right) = V'\left(\xi^P, \xi^{-P}\right) H'\left(\xi^P, \xi^{-P}\right) \Omega_{P,q}\left(\xi\right),$$

where H' and H are MLAs of M'^L and M^L, respectively, as in Theorem 3.3.8. Due to the uniqueness of decomposition (3.6–3.8) this leads to:

$$H'\left(\xi, \xi^{-1}\right) = V\left(\xi, \xi^{-1}\right) H\left(\xi, \xi^{-1}\right)$$

and

$$H\left(\xi, \xi^{-1}\right) = V'\left(\xi, \xi^{-1}\right) H'\left(\xi, \xi^{-1}\right),$$

meaning that the sets of annihilators of M^L and M'^L coincide or, equivalently, that M^L and M'^L are related by

$$M^L = M'^L G' \quad \text{and} \quad M'^L = M^L G,$$

where G' and G are suitable rational matrices. Hence,

$$M = M^L \Omega_{P,\ell} = M'^L G' \Omega_{P,\ell}$$

and

$$M' = M'^L \Omega_{P,\ell'} = M^L G \Omega_{P,\ell'}.$$

This leads to the following result.

Theorem 3.3.17 *Let M and M' be two P-PIR describing, respectively, the behaviors $\mathfrak{B}$ and $\mathfrak{B}'$. Then $\mathfrak{B} = \mathfrak{B}'$ if and only if there exist two rational matrices G and G' such that*

$$M\left(\xi, \xi^{-1}\right) = M'^L\left(\xi^P, \xi^{-P}\right) G'\left(\xi^P, \xi^{-P}\right) \Omega_{P,\ell}(\xi)$$

and

$$M'\left(\xi, \xi^{-1}\right) = M^L\left(\xi^P, \xi^{-P}\right) G\left(\xi^P, \xi^{-P}\right) \Omega_{P,\ell'}(\xi),$$

where M'^L and M^L are given as in (3.20). $\diamond$

Since P-PIR are a particular type of P-PLVR, it is natural to expect that they represent a more restricted class of systems. In fact, as we shall see in the next chapter, P-periodic systems with a P-PIR are precisely those which are controllable.

§3.4 Conclusion

This chapter is devoted to the study of P-periodic systems within the behavioral framework, as introduced by Margreet Kuijper and Jan C. Willems, in [47]. We used the *lifting technique* that establishes a one-to-one connection between the P-periodic system (Σ) and its associated lifted system (Σ^L), and took advantage of the properties induced by this lifting. A main ingredient was the (unique) decomposition

$R(\xi, \xi^{-1}) = R^L(\xi^P, \xi^{-P})\,\Omega_{P,q}(\xi)$, which provides a highway to link the representations of Σ and Σ^L, namely the P-PKR vs KR, P-PLVR vs LVR and P-PIR vs IR. An analogous of time-invariant behavior inclusion (equality) was extended to the P-periodic case. Further, a latent variable elimination procedure was obtained for P-PLVRs and P-PIRs.

"Discovery consists of seeing what everybody has seen and thinking what nobody has thought."
— Albert Szent-Gyorgyi

"There is no harm in doubt and skepticism, for it is through these that new discoveries are made."
— Richard Feynman

4

Controllability, autonomy and free variables

Using the definition of behavioral controllability, it is possible to obtain a correspondence between the controllability of a periodic system and of its associated lifted system. This is the key tool that, together with known results for the time-invariant case, enables us to characterize the controllability of periodic systems. The obtained results are applied to the particular case of periodic state space systems, namely in what concerns the relation between state space and behavioral controllability, leading to similar conclusions as for the time-invariant case. An autonomy characterization for periodic behaviors is obtained based on the connection established between the autonomy of a periodic system and the associated lifted system. We also prove the existence of an autonomous/controllable decomposition similar to what happens in the time-invariant case. Finally, we introduce a new concept of free variables and inputs, which can be regarded as a generalization of the one adopted for time-invariant systems, but appears to be more adequate for the periodic case.

§4.1 Controllability

We shall adopt the definition of behavioral controllability given in Definition 2.4.1, since, as mentioned earlier, it does not depend on the time invariant nature of the system. Recall that, according to this definition, a P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is behaviorally controllable if

$$\forall w_1, w_2 \in \mathfrak{B},\ \forall k_0 \in \mathbb{Z},\ \exists w \in \mathfrak{B},\ \exists k_1 \in \mathbb{Z}_+ \text{ s.t. } w(k) = \begin{cases} w_1(k), & k \le k_0 \\ w_2(k), & k > k_0 + k_1. \end{cases}$$

The next result allows us to link the controllability of a P-periodic system with the controllability of its associated lifted system.

Theorem 4.1.1 *A P-periodic system Σ is behaviorally controllable if and only if the associated lifted system Σ^L is behaviorally controllable.*

Proof.

($\Rightarrow$): Assume that Σ is controllable. Let $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ and $\widetilde{k}_0 \in \mathbb{Z}$. By construction there exist $w_1, w_2 \in \mathfrak{B}$ such that $Lw_i = \widetilde{w}_i,\ i = 1, 2$.

Take $k_0 := P\widetilde{k}_0 + P - 1$. Then, by the controllability of Σ, there exists $k_1 \in \mathbb{Z}_+$ and a trajectory $w \in \mathfrak{B}$ satisfying: $w(k) = w_1(k)$ for $k \leq k_0$ and $w(k) = w_2(k)$ for $k > k_0 + k_1$. Take $\widetilde{k}_1 = \lceil \frac{k_1}{P} \rceil + 1$.[4] Then, the trajectory $\widetilde{w} := Lw \in L\mathfrak{B}$ coincides with $\widetilde{w}_1$ for instants $k \leq \widetilde{k}_0$ and with $\widetilde{w}_2$ for instants $k > \widetilde{k}_0 + \widetilde{k}_1$, showing that $L\mathfrak{B}$ is controllable;

($\Leftarrow$): Assume now that Σ^L is controllable. Let $w_1, w_2 \in \mathfrak{B}$ and $k_0 \in \mathbb{Z}$. By construction there exist $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ such that $Lw_i = \widetilde{w}_i,\ i = 1, 2$.

Define $\widetilde{k}_0 = \lceil \frac{k_0 + 1}{P} \rceil - 1$. Since $L\mathfrak{B}$ is controllable, there exists $\widetilde{k}_1 \in \mathbb{Z}_+$ (which can clearly always taken to be not less than 1) and a trajectory $\widetilde{w} \in L\mathfrak{B}$ such that $\widetilde{w}(k) = \widetilde{w}_1(k)$ for $k \leq \widetilde{k}_0$ and $\widetilde{w}(k) = \widetilde{w}_2(k)$ for $k > \widetilde{k}_0 + \widetilde{k}_1$. Take $k_1 := P\left(\widetilde{k}_1 - 1\right) + 1 \geq 0$, and let $w := L^{-1}(\widetilde{w}) \in \mathfrak{B}$. Then, $w(k) = w_1(k)$ for $k \leq k_0$ and $w(k) = w_2(k)$ for $k > k_0 + k_1$, which proves that $\mathfrak{B}$ is controllable.

$\square$

In [47] an analogous result of Theorem 4.1.1 has been obtained for the alternative (time-invariant) twisted system.

The behavioral controllability characterization of time-invariant systems, given in [60, 61], together with Theorem 4.1.1, allows us to conclude the following.

Proposition 4.1.2 *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system with a P-PKR matrix R as in (3.5). Then Σ is controllable if and only if the corresponding matrix R^L (see (3.6) and (3.8)) is such that $R^L(\lambda, \lambda^{-1})$ has constant rank over $\mathbb{C} \setminus \{0\}$.* $\diamond$

[4] $\lceil \cdot \rceil$ represents the ceiling function, i.e., the integer round-up defined as $\lceil x \rceil = \min\{m \in \mathbb{Z} :\ m \geq x\}$.

Furthermore, if in addition the matrix $R^L(\xi,\xi^{-1}) \in \mathbb{R}^{g\times Pq}[\xi,\xi^{-1}]$ has full row rank, then the previous condition of $R^L(\lambda,\lambda^{-1})$ having constant rank over $\mathbb{C}\backslash\{0\}$ is equivalent to say that $R^L(\xi,\xi^{-1})$ is left-prime, i.e, all its left divisors are unimodular matrices in $\mathbb{R}^{g\times g}[\xi,\xi^{-1}]$. It appears that, due to the uniqueness of the decomposition (3.6–3.8), the left-primeness of $R^L(\xi,\xi^{-1})$ can be related to the following primeness property for $R(\xi,\xi^{-1})$.

Definition 4.1.3 *A Laurent-polynomial matrix $R(\xi,\xi^{-1}) \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$ with full row rank over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$ is said to be* left-prime over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$, *or simply* P-left-prime, *if whenever it is factored as*

$$R\left(\xi,\xi^{-1}\right) = D\left(\xi^P,\xi^{-P}\right)\overline{R}\left(\xi,\xi^{-1}\right),$$

with $D\left(\xi^P,\xi^{-P}\right) \in \mathbb{R}^{g\times g}\left[\xi^P,\xi^{-P}\right]$, then the factor $D\left(\xi^P,\xi^{-P}\right)$ is unimodular over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$ (or, equivalently, $D(\xi,\xi^{-1})$ is unimodular over $\mathbb{R}[\xi,\xi^{-1}]$). $\diamond$

Lemma 4.1.4 *Let $P \in \mathbb{N}$ and $R(\xi,\xi^{-1}) \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$ have full row rank over $\mathbb{R}\left[\xi^P,\xi^{-P}\right]$. Consider the associated matrix $R^L(\xi,\xi^{-1}) \in \mathbb{R}^{g\times Pq}[\xi,\xi^{-1}]$ according to the decomposition* (3.6–3.8)*. Then, the following conditions are equivalent:*

i) *$R^L(\xi,\xi^{-1})$ is left-prime;*

ii) *$R(\xi,\xi^{-1})$ is P-left-prime.*

Proof.

i) $\Rightarrow$ ii): Assume that $R(\xi,\xi^{-1})$ is not P-left-prime. Then there exist a non-unimodular square matrix $D\left(\xi^P,\xi^{-P}\right)$ and a matrix $\overline{R}(\xi,\xi^{-1}) \in \mathbb{R}^{g\times q}[\xi,\xi^{-1}]$ such that

$$R\left(\xi,\xi^{-1}\right) = D\left(\xi^P,\xi^{-P}\right)\overline{R}\left(\xi,\xi^{-1}\right).$$

Letting $\overline{R}^L(\xi,\xi^{-1})$ be the matrix corresponding to $\overline{R}(\xi,\xi^{-1})$ according to the decomposition (3.6–3.8), we have that

$$R\left(\xi,\xi^{-1}\right) = D\left(\xi^P,\xi^{-P}\right)\overline{R}^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}(\xi).$$

Due to the uniqueness of such decomposition, we conclude that

$$R^L\left(\xi^P,\xi^{-P}\right) = D\left(\xi^P,\xi^{-P}\right)\overline{R}^L\left(\xi^P,\xi^{-P}\right)$$

and hence

$$R^L\left(\xi,\xi^{-1}\right) = D\left(\xi,\xi^{-1}\right)\overline{R}^L\left(\xi,\xi^{-1}\right)$$

with $D(\xi,\xi^{-1})$ non-unimodular, showing that $R^L(\xi,\xi^{-1})$ is not left-prime;

ii) $\Rightarrow$ i): Assuming now that $R^L(\xi,\xi^{-1})$ is not left-prime, it is an easy exercise to check that all the previous reasoning can be reversed, due to the uniqueness of the property cited, allowing us to conclude that $R(\xi,\xi^{-1})$ is also not P-left-prime.

$\square$

This leads to the following direct characterization of controllability.

Theorem 4.1.5 *A P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$, with P-PKR, is controllable if and only if its minimal representation matrices $R(\xi,\xi^{-1})$ are P-left-prime.* $\diamond$

Since, for time-invariant behaviors, there is an equivalence between behavioral controllability and the existence of image representations (see [52]), Lemma 3.3.12, together with Theorem 4.1.1, allows us to prove the following result.

Theorem 4.1.6 *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system. Then $\mathfrak{B}$ has a P-PIR M if and only if $\mathfrak{B}$ is controllable.*

Proof. Due to the one-to-one correspondence between the image representations of $\mathfrak{B}$ and $L\mathfrak{B}$ we have the following scheme:

$$\begin{array}{ccc} \mathfrak{B} \text{ controllable} & \Longleftrightarrow & L\mathfrak{B} \text{ controllable} \\ & & \Updownarrow \\ \mathfrak{B} \text{ has a } P\text{-PIR} & \Longleftrightarrow & L\mathfrak{B} \text{ has an IR} \end{array}$$

$\square$

Example 4.1.7 *Consider the 2-periodic behavior of* Examples 3.1.2 *and* 3.3.13*, with P-PKR matrix $R(\xi,\xi^{-1}) = 1-\xi^{-1}$. According to* Theorem 4.1.5*, $\mathfrak{B}$ is controllable, since $R(\xi,\xi^{-1})$ is 2-left-prime, as the polynomial $1-\xi^{-1}$ has no non-trivial Laurent-polynomial factors in ξ^2. In fact, as we have seen in* Example 3.3.13*, $\mathfrak{B}$ allows a P-PIR.* $\diamond$

Remark 4.1.8 *This example shows that, contrary to time-invariant systems, a periodic behavior may be controllable without having free variables, i.e., without having variables whose values may be freely assigned on the whole time axis. The issue of free variables for periodic systems will be analysed later in* Section 4.4. $\diamond$

§4.2 Controllability of periodic state space systems

In this section we view a periodic state space system as a periodic behavioral system, study its controllability in behavioral terms and relate this property to the classical properties of state controllability and state reachability.

Note that the state space description (3.16) can be regarded as a particular case of a P-PKR. Indeed, letting $w := \begin{bmatrix} x^T & u^T & y^T \end{bmatrix}^T$, and due to the periodicity of matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$, the state space description

$$\begin{cases} (\sigma x)(k) = A(k)\,x(k) + B(k)\,u(k) \\ \quad y(k) = C(k)\,x(k) + D(k)\,u(k) \end{cases} \qquad k \in \mathbb{Z},$$

can be written as

$$\left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = 0, \quad t = 0, \ldots, P-1, \; k \in \mathbb{Z},$$

with

$$R_t\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi I_n - A(t) & -B(t) & 0 \\ -C(t) & -D(t) & I_p \end{bmatrix}$$

or still

$$\left(R\left(\sigma, \sigma^{-1}\right) w\right)(Pk) = 0, \quad k \in \mathbb{Z}, \tag{4.1}$$

with $R\left(\xi, \xi^{-1}\right)$ given by

$$\left[\begin{array}{ccc} \xi I_n - A(0) & -B(0) & 0 \\ -C(0) & -D(0) & I_p \\ \hline \xi\left(\xi I_n - A(1)\right) & -\xi B(1) & 0 \\ -\xi C(1) & -\xi D(1) & \xi I_p \\ \hline \vdots & \vdots & \vdots \\ \hline \xi^{P-1}\left(\xi I_n - A(P-1)\right) & -\xi^{P-1} B(P-1) & 0 \\ -\xi^{P-1} C(P-1) & -\xi^{P-1} D(P-1) & \xi^{P-1} I_p \end{array}\right].$$

Consequently, if $\mathfrak{B}$ is the behavior described by (4.1), the corresponding lifted behavior $L\mathfrak{B}$ is described by:

$$\left(R^L\left(\sigma, \sigma^{-1}\right)(Lw)\right)(k) = 0, \quad k \in \mathbb{Z},$$

with $R^L(\xi,\xi^{-1}) \in \mathbb{R}^{(n+p)P\times(n+m+p)P}[\xi,\xi^{-1}]$ given by

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
-A(0) & -B(0) & 0 & I_n & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
-C(0) & -D(0) & I_p & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -A(1) & -B(1) & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -C(1) & -D(1) & I_p & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \vdots & \vdots & \vdots \\
\xi I_n & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & -A(P-1) & -B(P-1) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & -C(P-1) & -D(P-1) & I_p
\end{array}\right].$$

In the sequel, for the sake of simplicity, we consider that $P = 2$, but our reasonings can easily be seen to apply to the general case. We then have

$$R^L\left(\xi,\xi^{-1}\right) = \left[\begin{array}{ccc|ccc}
-A(0) & -B(0) & 0 & I_n & 0 & 0 \\
-C(0) & -D(0) & I_p & 0 & 0 & 0 \\
\xi I_n & 0 & 0 & -A(1) & -B(1) & 0 \\
0 & 0 & 0 & -C(1) & -D(1) & I_p
\end{array}\right].$$

Now,

$$R^L\left(\xi,\xi^{-1}\right)\begin{bmatrix}
I_n & 0 & 0 & 0 & 0 & 0 \\
0 & I_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_p & 0 \\
0 & 0 & 0 & 0 & 0 & I_n \\
0 & 0 & I_m & 0 & 0 & 0 \\
0 & 0 & 0 & I_p & 0 & 0
\end{bmatrix}$$

$$= \left[\begin{array}{cccc|cc}
-A(0) & -B(0) & 0 & 0 & 0 & I_n \\
-C(0) & -D(0) & 0 & 0 & I_p & 0 \\
\hline
\xi I_n & 0 & -B(1) & 0 & 0 & -A(1) \\
0 & 0 & -D(1) & I_p & 0 & -C(1)
\end{array}\right].$$

Performing the block-row operations:

$$L_3 \leftarrow L_3 + A(1)\,L_1 \quad \text{and} \quad L_4 \leftarrow L_4 + C(1)\,L_1,$$

where L_i is the i^{th} block-row of R^L, we obtain the following matrix

$$\widetilde{R^L}\left(\xi,\xi^{-1}\right)=\left[\begin{array}{cccc|cc} -A(0) & -B(0) & 0 & 0 & 0 & I_n \\ -C(0) & -D(0) & 0 & 0 & I_p & 0 \\ \hline \multicolumn{4}{c|}{\widehat{R^L}\left(\xi,\xi^{-1}\right)} & \multicolumn{2}{c}{0} \end{array}\right],$$

with

$$\widehat{R^L}\left(\xi,\xi^{-1}\right)=\left[\begin{array}{ccc|c} \xi I_n - A(1)A(0) & -A(1)B(0) & -B(1) & 0 \\ \hline -C(1)A(0) & -C(1)B(0) & -D(1) & I_p \end{array}\right].$$

Recall now that, by Proposition 4.1.2, $\mathfrak{B}$ is behaviorally controllable if and only if $R^L\left(\lambda,\lambda^{-1}\right)$ has full row rank for all $\lambda\in\mathbb{C}\backslash\{0\}$.

Clearly the rank of the original matrix R^L coincides with the rank of $\widetilde{R^L}$. Moreover, due to its structure, $\widetilde{R^L}\left(\lambda,\lambda^{-1}\right)$ has a rank drop for some $\lambda\in\mathbb{C}\backslash\{0\}$ if and only if the same happens for the left upper block of $\widehat{R^L}\left(\lambda,\lambda^{-1}\right)$.

Thus,

$$\begin{aligned}\operatorname{rank}\widetilde{R^L}\left(\lambda,\lambda^{-1}\right) &= \operatorname{rank}\left[\begin{array}{ccc}\lambda I_n - A(1)A(0) & -A(1)B(0) & -B(1)\end{array}\right]\\ &= \operatorname{rank}\left[\begin{array}{ccc}\lambda I_n - A(1)A(0) & B(1) & A(1)B(0)\end{array}\right] = \operatorname{rank}\left[\begin{array}{cc}\lambda I_n - A_0 & B_0\end{array}\right],\end{aligned}$$

with $A_0,\ B_0$ as in (1.2), (1.3), respectively, that is,

$$A_0 = A(1)A(0)$$

$$B_0=\left[\begin{array}{cc}B(1) & A(1)B(0)\end{array}\right].$$

Therefore $\mathfrak{B}$ is behaviorally controllable if and only if

$$\operatorname{rank}\left[\begin{array}{cc}\lambda I_n - A_0 & B_0\end{array}\right]=n,\quad \forall\lambda\in\mathbb{C}\backslash\{0\}.$$

Suppose now that for some $\lambda^*\in\mathbb{C}\backslash\{0\}$, $\operatorname{rank}\left[\begin{array}{cc}\lambda^* I_n - A_0 & B_0\end{array}\right]<n$. This means that there exists $0\neq v^*\in\mathbb{R}^{1\times n}$ such that

$$v^*\left[\begin{array}{ccc}\lambda^* I_n - A(1)A(0) & B(1) & A(1)B(0)\end{array}\right]=0,$$

which is equivalent to

$$v^* \left(\lambda^* I_n - A(1) A(0)\right) = 0;^{5} \tag{4.2}$$

$$v^* B(1) = 0; \tag{4.3}$$

$$v^* A(1) B(0) = 0. \tag{4.4}$$

Consequently, the product

$$v^* A(1) \begin{bmatrix} \lambda^* I_n - A_1 & B_1 \end{bmatrix},$$

where

$$A_1 = A(0) A(1)$$

$$B_1 = \begin{bmatrix} B(0) & A(0) B(1) \end{bmatrix},$$

is given by:

$$\begin{aligned} & v^* A(1) \begin{bmatrix} \lambda^* I_n - A(0) A(1) & B(0) & A(0) B(1) \end{bmatrix} \\ & = \begin{bmatrix} \lambda^* v^* A(1) - v^* A(1) A(0) A(1) & \underbrace{v^* A(1) B(0)}_{=0,\ \text{by (4.4)}} & \underbrace{v^* A(1) A(0)}_{=\lambda^* v^*,\ \text{by (4.2)}} B(1) \end{bmatrix} \\ & = \begin{bmatrix} \underbrace{\left(\lambda^* v^* - v^* A(1) A(0)\right)}_{=0,\ \text{by (4.2)}} A(1) & 0 & \lambda^* \underbrace{v^* B(1)}_{=0,\ \text{by (4.3)}} \end{bmatrix} = 0. \end{aligned}$$

Since $v^* A(1) \neq 0$ (otherwise $v^* A(1) A(0) = 0$ and v^* would be a left eigenvector of $A(1) A(0)$ associated to the eigenvalue zero, which is not the case since, by (4.2), v^* is a left eigenvector of $A(1) A(0)$ associated to $\lambda^* \neq 0$), we conclude that

$$\operatorname{rank} \begin{bmatrix} \lambda^* I_n - A_1 & B_1 \end{bmatrix} < n.$$

Taking into account that this reasoning can be reversed, we obtain that

$$\begin{aligned} & \left\{ \operatorname{rank} \begin{bmatrix} \lambda I_n - A_0 & B_0 \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \backslash \{0\} \right\} \\ & \qquad \Leftrightarrow \left\{ \operatorname{rank} \begin{bmatrix} \lambda I_n - A_1 & B_1 \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \backslash \{0\} \right\}. \end{aligned}$$

Noting that this reasoning can be easily extended to the general P-periodic case, yields the next result.

[5] i.e., v^* is a left eigenvector of $A(1) A(0)$.

Theorem 4.2.1 *Let Σ be a P-periodic state space system, described as in* (1.1) *and* (3.16)*, and let $\Sigma_t = (A_t, B_t, C_t, D_t)$ be the P time-invariant systems obtained by the invariant dynamical decomposition, described in* Section 1.2*. Then the following conditions are equivalent:*

i) *The behavior $\mathfrak{B}$ of Σ is behaviorally controllable;*

ii) $\operatorname{rank}\begin{bmatrix} \lambda I_n - A_t & B_t \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \backslash \{0\}$, *for at least one t in $\{0, \ldots, P-1\}$;*

iii) $\operatorname{rank}\begin{bmatrix} \lambda I_n - A_t & B_t \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \backslash \{0\}$, *for all t in $\{0, \ldots, P-1\}$.*

$\diamond$

Combining Theorems 1.3.3, 1.3.10 and 4.2.1, we are able to relate behavioral controllability with the classical property of complete state controllability.

Theorem 4.2.2 *The behavior $\mathfrak{B}$ of a P-periodic state space system Σ is (behaviorally) controllable if and only if Σ is completely state controllable.* $\diamond$

Combining Theorems 4.2.2 and 1.3.13, we obtain the following result.

Corollary 4.2.3 *A P-periodic state space system is completely state reachable if and only if it is completely state trim and its behavior is controllable.* $\diamond$

This generalizes a similar result obtained in [61] for the time-invariant case.

§4.3 Autonomy

The Definition 2.4.5 of autonomy presented in Chapter 2 applies to general behaviors, and hence also to the P-periodic case. We recall this definition in order to facilitate the reading of the remaining within this chapter.

Definition 4.3.1 [47] *The P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ is said to be autonomous if for all $k_0 \in \mathbb{Z}$ and all $w_1, w_2 \in \mathfrak{B}$*

$$w_1(k) = w_2(k) \text{ for } k < k_0 \quad \Rightarrow \quad w_1 = w_2.$$

$\diamond$

Similar to what happens with controllability, the autonomy of $\mathfrak{B}$ and of $L\mathfrak{B}$ are one-to-one related.

Theorem 4.3.2 [47] *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system. Then $\mathfrak{B}$ is autonomous if and only if $L\mathfrak{B}$ is autonomous.* $\diamond$

Taking into account the characterization of autonomy for time-invariant behaviors given in Theorem 2.4.6, the following result is easily obtained.

Corollary 4.3.3 [47] *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system given by a P-PKR matrix R. Then $\mathfrak{B}$ is autonomous if and only if the corresponding representation matrix of the associated lifted system, R^L, has full column rank.* $\diamond$

We now prove a similar result to Theorem 2.4.7 on the autonomous/controllable decomposition of periodic behaviors.

Theorem 4.3.4 *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be P-periodic. Then there exist $\mathfrak{B}^{\mathrm{a}}, \mathfrak{B}^{\mathrm{c}} \subset \mathfrak{B}$ such that:*

i) $\mathfrak{B}^{\mathrm{a}}$ *is autonomous;*

ii) $\mathfrak{B}^{\mathrm{c}}$ *is controllable;*

iii) $\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}$.

Proof. Define $\widetilde{\mathfrak{B}} := L\mathfrak{B}$. Then, there exist sub-behaviors $\widetilde{\mathfrak{B}}^{\mathrm{a}}$ and $\widetilde{\mathfrak{B}}^{\mathrm{c}}$ such that

$$\widetilde{\mathfrak{B}} = \widetilde{\mathfrak{B}}^{\mathrm{a}} \oplus \widetilde{\mathfrak{B}}^{\mathrm{c}}.$$

As stated in Proposition 3.1.3, L is a homeomorphism and therefore its inverse L^{-1} is well defined. Define $\mathfrak{B}^{\mathrm{a}} := L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{a}}$ and $\mathfrak{B}^{\mathrm{c}} := L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{c}}$. Note that, by Theorem 4.3.2, $\mathfrak{B}^{\mathrm{a}}$ is autonomous and $\mathfrak{B}^{\mathrm{c}}$ is controllable due to Theorem 4.1.1.

Since L is a homeomorphism, we then have that:

$$\mathfrak{B}^{\mathrm{a}} \cap \mathfrak{B}^{\mathrm{c}} = L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{a}} \cap L^{-1}\widetilde{\mathfrak{B}}^{\mathrm{c}} = L^{-1}\left(\widetilde{\mathfrak{B}}^{\mathrm{a}} \cap \widetilde{\mathfrak{B}}^{\mathrm{c}}\right) = L^{-1}\left(\{0\}\right) = \{0\}.$$

Finally take $w \in \mathfrak{B}$. Let $\widetilde{w} := L(w)$ and take $\widetilde{w}_a$ and $\widetilde{w}_c$ to be such that

$$\widetilde{w} = \widetilde{w}_a + \widetilde{w}_c, \quad \widetilde{w}_a \in \widetilde{\mathfrak{B}}^{\mathrm{a}},\ \widetilde{w}_c \in \widetilde{\mathfrak{B}}^{\mathrm{c}}.$$

Define $w_a := L^{-1}(\widetilde{w}_a)$, $w_c := L^{-1}(\widetilde{w}_c)$. Then,

$$w = L^{-1}(\widetilde{w}) = L^{-1}(\widetilde{w}_a + \widetilde{w}_c) = L^{-1}(\widetilde{w}_a) + L^{-1}(\widetilde{w}_c) = w_a + w_c.$$

Thus we conclude that $\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}}$. $\square$

Example 4.3.5 *Let* $\Sigma = (\mathbb{Z}, \mathbb{R}^2, \mathfrak{B})$ *be the* 2*-periodic system with representation matrix*

$$R\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi^2 - 1 & \xi^4 - 1 \end{bmatrix} = R^L\left(\xi^2, \xi^{-2}\right)\Omega_{2,2}(\xi),$$

with $R^L(\xi, \xi^{-1}) = \left[\begin{array}{cc|cc} \xi - 1 & \xi^2 - 1 & 0 & 0 \end{array}\right]$*. It can be shown that the time-invariant lifted behavior* $L\mathfrak{B}$*, represented by* $R^L(\xi, \xi^{-1})$*, has the following autonomous/controllable decomposition:*

$$L\mathfrak{B} = (L\mathfrak{B})^{\mathrm{a}} \oplus (L\mathfrak{B})^{\mathrm{c}},$$

where the autonomous sub-behavior $(L\mathfrak{B})^{\mathrm{a}}$ *is represented by*

$$R_a^L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi - 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the controllable sub-behavior $(L\mathfrak{B})^{\mathrm{c}}$ *has a representation*

$$R_c^L\left(\xi, \xi^{-1}\right) = \left[\begin{array}{cc|cc} 1 & \xi + 1 & 0 & 0 \end{array}\right].$$

Further details on how to obtain such a decomposition for time-invariant systems can be found, for instance, in [52]*. This implies that* $\mathfrak{B}$ *can be decomposed as*

$$\mathfrak{B} = \mathfrak{B}^{\mathrm{a}} \oplus \mathfrak{B}^{\mathrm{c}},$$

with $\mathfrak{B}^{\mathrm{a}} = L^{-1}\left((L\mathfrak{B})^{\mathrm{a}}\right)$ *represented by*

$$R_a\left(\xi, \xi^{-1}\right) = R_a^L\left(\xi^2, \xi^{-2}\right)\Omega_{2,2}(\xi) = \begin{bmatrix} \xi^2 - 1 & 0 \\ 0 & 1 \\ \xi & 0 \\ 0 & \xi \end{bmatrix}$$

and $\mathfrak{B}^{\mathrm{c}} = L^{-1}\left((L\mathfrak{B})^{\mathrm{c}}\right)$ *represented by*

$$R_c\left(\xi, \xi^{-1}\right) = R_c^L\left(\xi^2, \xi^{-2}\right)\Omega_{2,2}(\xi) = \begin{bmatrix} 1 & \xi^2 + 1 \end{bmatrix}.$$

$\diamond$

§4.4 Free variables and P-periodic inputs

Recall that, as mentioned in Section 2.4, in the time-invariant case the properties of controllability and autonomy are, respectively, related to the existence or the absence of free variables. As the next examples show, this no longer holds in the P-periodic case.

Example 4.4.1 *Consider again the* 2*-periodic behavior of* Examples 3.1.2, 3.3.13 *and* 4.1.7*. As shown in* Example 4.1.7, $\mathfrak{B}$ *is controllable. However,* $\mathfrak{B}$ *has no free variables, since the values of* w *on each odd time instant* $(2k+1)$ *and its consecutive one* $(2k+2)$ *must coincide.* $\diamond$

Example 4.4.2 *Let* $\mathfrak{B} \subset \mathbb{R}$ *be the* 2*-periodic behavior described by* $w\,(2k) = 0,\ k \in \mathbb{Z}$. *Clearly the only system variable* w *is not free, since it is required to be zero on the even time instants. However,* $\mathfrak{B}$ *is not autonomous. Indeed fixing the values of* $w\,(k)$ *for* $k \leq 0$ *does not yield a unique trajectory, since the values of* $w\,(2k+1)\,,\ k \geq 0$ *can still be chosen freely. Thus the absence of free variables does not imply autonomy.* $\diamond$

These two examples suggest that a more sophisticated notion of free variable should be considered in the P-periodic case.

Definition 4.4.3 *Let* $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ *be a* P*-periodic behavior in* q *variables. The* i^{th} *system variable* $w_i,\ i \in \{1,\ldots,q\}$*, is said to be* P-periodically free with offset t *or* t-P-periodically free*, for* $t = 0,\ldots,P-1$*, if* $w_i\,(Pk+t)\,,\ k \in \mathbb{Z}$*, is not restricted by the behavior. More precisely, if for all* $\alpha \in \mathbb{R}^{\mathbb{Z}}$*, there exists* $w^* \in \mathfrak{B}$ *such that its* i^{th}*-component satisfies*

$$w_i^*\,(Pk+t) = \alpha\,(k)\,, \quad k \in \mathbb{Z}.$$

Moreover, w_i *is said to be just* P-periodically free *if it is* P-periodically free with offset t*, for some* $t = 0,\ldots,P-1$. $\diamond$

Note that, regarding time-invariance as 1-periodicity, Definition 4.4.3 yields the usual definition of free variable for time-invariant behaviors.

The following lemma is a direct consequence of Definition 4.4.3.

Lemma 4.4.4 *Given a* P*-periodic behavior* $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$*, the* i^{th} *system variable* $w_i,\ i \in \{1,\ldots,q\}$*, is* t*-*P*-periodically free* (in $\mathfrak{B}$)*, for* $t = 0,\ldots,P-1$*, if and only if* $(Lw)_{tq+i}$ *is free in* $L\mathfrak{B}$. $\diamond$

It is now easy to conclude that a controllable P-periodic behavior must have P-periodically free variables.

Example 4.4.5 *Recall* Example 4.4.1. *As we have seen there, w is not free. However, w is 2-periodically free. To see this, recall that the associated lifted behavior $L\mathfrak{B}$ is described by*

$$\left(\left[\begin{array}{cc} 1 & -\sigma^{-1} \end{array}\right]\left[\begin{array}{c} \widetilde{w}_1 \\ \widetilde{w}_2 \end{array}\right]\right)(k) = 0, \quad k \in \mathbb{Z},$$

or, equivalently,

$$\widetilde{w}_1(k) = \widetilde{w}_2(k-1), \quad k \in \mathbb{Z},$$

showing that either $\widetilde{w}_1$ or $\widetilde{w}_2$ are free in $L\mathfrak{B}$. Thus w is 2-periodically free since it is 2-periodically free with offsets $t = 0$ or $t = 1$. $\diamond$

As for autonomy, the following characterization in terms of P-periodically free variables holds.

Theorem 4.4.6 *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system. Then $\mathfrak{B}$ is autonomous if and only if $\mathfrak{B}$ has no P-periodically free variables.*

Proof. Recall that, by Theorem 4.3.2, $\mathfrak{B}$ is autonomous if and only if $L\mathfrak{B}$ is autonomous. In turn, the lifted system is autonomous if and only if $L\mathfrak{B}$ has no free variables which, by Lemma 4.4.4, is equivalent to the autonomy of $\mathfrak{B}$. $\square$

Example 4.4.7 *Recall* Example 4.4.2. *As we have seen there, although $\mathfrak{B}$ is not autonomous, the system variable w is not free. Notice, however, that w is 2-periodically free since, in this case, we have*

$$R\left(\xi, \xi^{-1}\right) = 1 = \left[\begin{array}{cc} 1 & 0 \end{array}\right]\left[\begin{array}{c} 1 \\ \xi \end{array}\right],$$

which leads to

$$R^L\left(\xi, \xi^{-1}\right) = \left[\begin{array}{cc} 1 & 0 \end{array}\right].$$

Therefore the associated lifted behavior $L\mathfrak{B}$ is described by

$$\left(R^L\left(\sigma, \sigma^{-1}\right)\left[\begin{array}{c} \widetilde{w}_1 \\ \widetilde{w}_2 \end{array}\right]\right)(k) = 0, \quad k \in \mathbb{Z},$$

or, equivalently,

$$\widetilde{w}_1(k) = 0, \quad k \in \mathbb{Z}.$$

Thus $\widetilde{w}_2$ *is free and* w *is* 2*-periodically free since it is* 2*-periodically free with offset* $t = 1$*.* ◇

Remark 4.4.8 *The absence of* P*-periodically free variables implies the absence of free variables in* $\mathfrak{B}$ *which, in turn, is equivalent to a full column rank condition on* R (*this can be shown by similar arguments as used in* [52] *for the time-invariant case*)*. Thus the full column rank condition on* R *is only a necessary condition for autonomy, but not a sufficient one* (*as illustrated in* Example 4.4.2*, where* $R(\sigma, \sigma^{-1}) = 1$)*. This can also be seen* (*using* Corollary 4.3.3) *from the fact that* R^L *has full column rank implies that* R *has full column rank, but not vice-versa.*[6] ◇

In the time-invariant case, an input is defined as a maximally free set of system variables, i.e., as a set of variables which are simultaneously free and which, once fixed, leave no extra free variables in the system. When defining simultaneously free components of a trajectory, in a P-periodic behavior, one has to take into account that such components may be P-periodically free with different offsets. This is illustrated in the following example.

Example 4.4.9 *Let* $\mathfrak{B} \subset (\mathbb{R}^2)^{\mathbb{Z}}$ *be the* 3*-periodic behavior given by the equations*

$$w_2(3k) = w_2(3k+1) = w_1(3k+2) = 0, \quad k \in \mathbb{Z}.$$

Clearly the values of $w_1(3k)$*,* $w_1(3k+1)$ *and* $w_2(3k+2)$*,* ($k \in \mathbb{Z}$)*, are free, i.e.,* w_1 *is* 3*-periodically free with offsets* 0 *and* 1*, and* w_2 *is* 3*-periodically free with offset* 2*. Note further, that none of the variables is free in all the possible offsets* $t = 0, 1, 2$*, i.e., there is no variable whose values can be freely assigned at all time instants. This can be put in a more compact form by saying that* (w_1, w_1, w_2) *is* $(0, 1, 2)$*-*3*-periodically free. Note that, in this case, the freeness in the system cannot be assigned to one of the two system variables alone. Therefore, neither* w_1 *nor* w_2 *can be taken as an "input", in the classical, time-invariant sense. This suggests using an alternative approach.*

[6] See Appendix B.

Using the operator $\Omega_{P,q}$ *introduced in* Section 3.2, *we have that*

$$\Omega_{3,2}(\sigma)(w) = \begin{bmatrix} w_1 \\ w_2 \\ \sigma w_1 \\ \sigma w_2 \\ \sigma^2 w_1 \\ \sigma^2 w_2 \end{bmatrix}.$$

Thus

$$\begin{aligned} w_1(3k) &= (\Omega_{3,2}(\sigma) w)_1 (3k) \\ w_1(3k+1) &= (\Omega_{3,2}(\sigma) w)_3 (3k) \\ w_2(3k+2) &= (\Omega_{3,2}(\sigma) w)_6 (3k), \end{aligned}$$

where the sub-indices correspond to the components of $\Omega_{3,2}(\sigma) w$. *Now*

$$u = \begin{bmatrix} (\Omega_{3,2}(\sigma) w)_1 & (\Omega_{3,2}(\sigma) w)_3 & (\Omega_{3,2}(\sigma) w)_6 \end{bmatrix}^T$$

is a free set of variables of $\Omega_{3,2}(\sigma) w$, *since* $u(3k)$ *can be chosen freely for all* $k \in \mathbb{Z}$, *i.e., given* $\alpha \in (\mathbb{R}^3)^{\mathbb{Z}}$, *there exists* $w^* \in \mathfrak{B}$, *such that,*

$$u^*(3k) = \begin{bmatrix} (\Omega_{3,2}(\sigma) w^*)_1 & (\Omega_{3,2}(\sigma) w^*)_3 & (\Omega_{3,2}(\sigma) w^*)_6 \end{bmatrix}^T (3k) = \alpha(k), \ k \in \mathbb{Z}.$$

Moreover, u *is a* maximally *free set of variables, in the sense that once* u *is fixed (say,* $u(3k) = 0$, $k \in \mathbb{Z}$*) no other free components are left in* $\Omega_{3,2}(\sigma) w$. *Therefore, we call* u *a* P-periodic input *of* $\mathfrak{B}$. *The complementary components of* $\Omega_{3,2}(\sigma) w$,

$$y = \begin{bmatrix} (\Omega_{3,2}(\sigma) w)_2 & (\Omega_{3,2}(\sigma) w)_4 & (\Omega_{3,2}(\sigma) w)_5 \end{bmatrix}^T,$$

constitute the corresponding P*-periodic output.* $\diamond$

In the general case, given a P-periodic behavior $\mathfrak{B}$ with variable w, a choice of (possibly repeated) components of w, $\begin{bmatrix} w_{i_1} & \cdots & w_{i_m} \end{bmatrix}^T$, $i_r \in \{1, \ldots, q\}$ for $r = 1, \ldots, m$, is said to be $(t_1, \ldots, t_m)$-*P-periodically free*, $t_r \in \{0, \ldots, P-1\}$ for $r = 1, \ldots, m$, if for all $\alpha_r \in \mathbb{R}^{\mathbb{Z}}$, there exists $w^* \in \mathfrak{B}$ such that its ${i_r}^{th}$-component satisfies

$$w^*_{i_r}(Pk + t_r) = \alpha_r(k), \quad k \in \mathbb{Z}.$$

Note that $\begin{bmatrix} w_{i_1} & \cdots & w_{i_m} \end{bmatrix}^T$ is $(t_1, \ldots, t_m)$-*P-periodically free* if and only if

$$u = \begin{bmatrix} (\Omega_{P,q}(\sigma) w)_{t_1 q + i_1} & \cdots & (\Omega_{P,q}(\sigma) w)_{t_m q + i_m} \end{bmatrix}^T$$

is a free set of variables of $\Omega_{P,q}(\sigma) w$, with $\Omega_{P,q}(\xi)$ defined as in (3.7).

Definition 4.4.10 *Given a P-periodic behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ with variable $\begin{bmatrix} w_1 & \cdots & w_q \end{bmatrix}^T$, a choice of components*

$$u = \begin{bmatrix} (\Omega_{P,q}(\sigma) w)_{\ell_1} & \cdots & (\Omega_{P,q}(\sigma) w)_{\ell_m} \end{bmatrix}^T$$

of $\Omega_{P,q}(\sigma) w$ is said to be a P-periodic input of $\mathfrak{B}$ if u is a maximally free set of variables of $\Omega_{P,q}(\sigma) w$ in the following sense:

i) *u is free, i.e., $\forall \alpha \in (\mathbb{R}^m)^{\mathbb{Z}}$ $\exists w^* \in \mathfrak{B}$ such that*

$$u^*(Pk) = \begin{bmatrix} (\Omega_{P,q}(\sigma) w^*)_{\ell_1} & \cdots & (\Omega_{P,q}(\sigma) w^*)_{\ell_m} \end{bmatrix}^T (Pk) = \alpha(k), \ k \in \mathbb{Z};$$

ii) *The set of trajectories*

$$\{(\Omega_{P,q}(\sigma) w)(Pk), \ w \in \mathfrak{B} : \ u(Pk) = 0\}$$

has no free variables.

In this case the remaining components, y, of $\Omega_{P,q}(\sigma) w$ are said to constitute a P-periodic output of $\mathfrak{B}$. Finally, an input-output structure for $\mathfrak{B}$ is defined as a partition (u, y) of the components of $\Omega_{P,q}(\sigma) w$, such that u is an input and y is an output. $\diamond$

Remark 4.4.11 *The fact that*

$$u = \begin{bmatrix} (\Omega_{P,q}(\sigma) w)_{\ell_1} & \cdots & (\Omega_{P,q}(\sigma) w)_{\ell_m} \end{bmatrix}^T$$

is a P-periodic input means that the values of

$$w_{i_r}(Pk + t_r), \ i_r \in \{1, \ldots, q\}, \ r = 1, \ldots, m,$$

where $l_r = t_r q + i_r$, may be freely assigned. $\diamond$

Noticing that

$$(\Omega_{P,q}(\sigma) w)(Pk) = (Lw)(k), \quad k \in \mathbb{Z},$$

leads easily to the following result.

Proposition 4.4.12 *$u = \begin{bmatrix} (\Omega_{P,q}(\sigma) w)_{\ell_1} & \cdots & (\Omega_{P,q}(\sigma) w)_{\ell_m} \end{bmatrix}^T$ is a P-periodic input of $\mathfrak{B}$ if and only if*

$$\widetilde{u} = \begin{bmatrix} (Lw)_{\ell_1} & \cdots & (Lw)_{\ell_m} \end{bmatrix}^T$$

is an input of the time-invariant behavior $L\mathfrak{B}$. $\diamond$

Given this established relationship between the P-periodically free variables of a P-periodic behavior and the free variables of its associated lifted system, it is now possible to define input-output structures in the periodic case based on the available results for time-invariant systems. This leads to the following result.

Theorem 4.4.13 *Every P-periodic behavior $\mathfrak{B}$ admits an input-output structure.* $\diamond$

Example 4.4.14 *Consider the 3-periodic behavior $\mathfrak{B}$ with P-PKR matrix*

$$R\left(\xi,\xi^{-1}\right)=\begin{bmatrix}\xi-1 & 2\xi^{-1}\\ \xi^2+\xi & 2\xi^2\\ 1 & \xi^2-\xi\\ 2\xi^3+\xi^2 & 2\xi^3-\xi^2\end{bmatrix}.$$

Its associated lifted system has also a kernel representation, that is,

$$L\mathfrak{B}=\ker R^L\left(\sigma,\sigma^{-1}\right),$$

with

$$R^L\left(\xi,\xi^{-1}\right)=\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 2\xi^{-1}\\ 0 & 0 & 1 & 0 & 1 & 2\\ 1 & 0 & 0 & -1 & 0 & 1\\ 2\xi & 2\xi & 0 & 0 & 1 & -1\end{bmatrix}.$$

Letting

$$\widetilde{R}^L\left(\xi,\xi^{-1}\right)=R^L\left(\xi,\xi^{-1}\right)\begin{bmatrix}1&0&0&0&0&0\\0&0&0&0&1&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{bmatrix}$$

$$=\left[\begin{array}{cccc|cc}-1 & 1 & 0 & 0 & 0 & 2\xi^{-1}\\ 0 & 1 & 0 & 1 & 0 & 2\\ 1 & 0 & -1 & 0 & 0 & 1\\ 2\xi & 0 & 0 & 1 & 2\xi & -1\end{array}\right]=:\left[\begin{array}{c|c} P\left(\xi,\xi^{-1}\right) & -Q\left(\xi,\xi^{-1}\right)\end{array}\right],$$

the lifted system can be represented as

$$\left(P\left(\sigma,\sigma^{-1}\right)\begin{bmatrix}(Lw)_1\\(Lw)_3\\(Lw)_4\\(Lw)_5\end{bmatrix}\right)(k)=\left(Q\left(\sigma,\sigma^{-1}\right)\begin{bmatrix}(Lw)_2\\(Lw)_6\end{bmatrix}\right)(k),\quad k\in\mathbb{Z}.$$

Since $\det P(\xi, \xi^{-1}) \neq 0$*, the previous equation is solvable for*

$$\begin{bmatrix} (Lw)_1 \\ (Lw)_3 \\ (Lw)_4 \\ (Lw)_5 \end{bmatrix},$$

given any arbitrary values of $\widetilde{u} := \begin{bmatrix} Lw_2 & Lw_6 \end{bmatrix}^T$*; moreover, the set of trajectories corresponding to* $\widetilde{u} \equiv 0$ *has no free variables. Thus* $\widetilde{u}$ *is an input in* $L\mathfrak{B}$ *and, consequently*

$$u = \begin{bmatrix} (\Omega_{3,2}(\sigma)\, w)_2 & (\Omega_{3,2}(\sigma)\, w)_6 \end{bmatrix}^T$$

is a 3*-periodic input for* $\mathfrak{B}$*.* ◇

§4.5 Conclusion

The behavioral notion of controllability has been considered for P-periodic behaviors. We proved that a P-periodic behavior $\mathfrak{B}$ is controllable if and only if its associated lifted behavior $L\mathfrak{B}$ is controllable. Based on this result, we were able to obtain a characterization for testing controllability that corresponds to a particular primeness property of the P-PKR matrix. We also proved that controllable P-periodic behaviors are exactly the ones which allow a P-PIR. These results were then applied to P-periodic state space systems allowing us to relate behavioral controllability with complete state controllability as it was defined in Chapter 2. Furthermore, we considered the behavioral notions of autonomy and managed to prove that any P-periodic behavior allows an autonomous/controllable decomposition similar to what happens in the time-invariant case.

Finally, we introduced a new concept of free variable, designated as P-periodic free variable, which allowed us to relate autonomy and absence of free variables. The introduction of this new definition for freeness, which generalizes the definition of free variable in the time-invariant case, in P-periodic behaviors made possible the definition of P-periodic inputs, allowing us to establish an input-output structure.

"Success is the ability to go from one failure to another with no loss of enthusiasm."

— Sir Winston Churchill

"There are no such things as applied sciences, only applications of science."

— Louis Pasteur

5 Reconstructibility and observability

Using the definition of behavioral reconstructibility stated in Chapter 2, we obtain a correspondence between the reconstructibility of a periodic system and the reconstructibility of its associated lifted system. This is the key tool that enables the characterization of reconstructibility of periodic systems, by using known results for the time-invariant case. These results are applied to the particular case of periodic state space systems, in order to analyse the relationship between the behavioral and the classical reconstructibility notions, leading to similar characterizations as for the case of time-invariant state space systems. Further, we prove the equivalence between the notions of *Willems-observability* and reconstructibility for periodic behaviors, as happens for time-invariant behaviors.

§5.1 Reconstructibility

The adopted definition of behavioral reconstructibility is given by Definition 2.5.1. Recall that, according to this definition, in a P-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}, \mathfrak{B})$, whose system variable w is partitioned as $w = (w_1, w_2)$, and given $\delta \geq 0$, w_2 is said to be δ-reconstructible from w_1 if

$$\left\{ w_1\Big|_{[k_0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ w_2\Big|_{[k_0+\delta,+\infty)} \equiv 0 \right\}, \quad \forall k_0 \in \mathbb{Z}.$$

Theorem 5.1.1 *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}, \mathfrak{B})$ be a P-periodic system whose system variable w is partitioned as $w = (w_1, w_2)$. Then w_2 is* reconstructible *from w_1 if and only* *if in the associated lifted system,* $\Sigma^L = \left(\mathbb{Z}, \mathbb{R}^{Pq_1} \times \mathbb{R}^{Pq_2}, L\mathfrak{B}\right)$*,* Lw_2 *is* reconstructible *from* Lw_1*.*

Proof.

($\Rightarrow$): Assume that w_2 is δ-*reconstructible* from w_1, for some $\delta \geq 0$, i.e., that

$$\left\{ w_1\Big|_{[k_0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ w_2\Big|_{[k_0+\delta,+\infty)} \equiv 0 \right\}, \quad \forall k_0 \in \mathbb{Z}. \tag{5.1}$$

Consider $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ and $\widetilde{k}_0 \in \mathbb{Z}$. Let $w_1, w_2 \in \mathfrak{B}$ be such that $Lw_i = \widetilde{w}_i$, $i = 1, 2$.

Define $k_0 := P\widetilde{k}_0$. By the reconstructibility of Σ, there exists $\delta \geq 0$ such that (5.1) holds. Take $\widetilde{\delta} = \lceil \dfrac{\delta}{P} \rceil$. Then, $\forall \widetilde{k}_0 \in \mathbb{Z}$,

$$\begin{aligned}
\left\{ \widetilde{w}_1\Big|_{\left[\widetilde{k}_0,+\infty\right)} \equiv 0 \right\} &\Rightarrow \left\{ \begin{bmatrix} w_1(Pk) \\ \vdots \\ w_1(Pk+(P-1)) \end{bmatrix} = 0, \quad \forall k \in \left[\widetilde{k}_0, +\infty\right) \right\} \\
&\Rightarrow \left\{ w_1(\ell) = 0, \quad \forall \ell \in \left[P\widetilde{k}_0, +\infty\right) \right\} \\
&\Leftrightarrow \{ w_1(\ell) = 0, \quad \forall \ell \in [k_0, +\infty) \} \\
&\Rightarrow \{ w_2(\ell) = 0, \quad \forall \ell \in [k_0+\delta, +\infty) \} \\
&\Rightarrow \left\{ \begin{bmatrix} w_2(Pk) \\ \vdots \\ w_2(Pk+(P-1)) \end{bmatrix} = 0, \quad \forall k \geq \frac{k_0+\delta}{P} \right\} \\
&\Rightarrow \left\{ \begin{bmatrix} w_2(Pk) \\ \vdots \\ w_2(Pk+(P-1)) \end{bmatrix} = 0, \quad \forall k \in \left[\widetilde{k}_0+\widetilde{\delta}, +\infty\right) \right\} \\
&\Rightarrow \left\{ \widetilde{w}_2\Big|_{\left[\widetilde{k}_0+\widetilde{\delta},+\infty\right)} \equiv 0 \right\},
\end{aligned}$$

showing that Lw_2 is $\widetilde{\delta}$-*reconstructible* from Lw_1;

($\Leftarrow$): Assume now that Lw_2 is $\widetilde{\delta}$*-reconstructible* from Lw_1, for some $\widetilde{\delta} \geq 0$. Let $w_1, w_2 \in \mathfrak{B}$ and $k_0 \in \mathbb{Z}$. Let further $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ such that $Lw_i = \widetilde{w}_i,\ i = 1, 2$.

Then,

$$\{w_1(k) = 0, \quad \forall k \in [k_0, +\infty)\} \Rightarrow \left\{ \begin{bmatrix} w_1(P\ell) \\ \vdots \\ w_1(P\ell + (P-1)) \end{bmatrix} = 0, \quad \forall \ell \geq \frac{k_0}{P} \right\}$$

$$\Rightarrow \left\{ \widetilde{w}_1(\ell) = 0, \quad \forall \ell \geq \lceil \frac{k_0}{P} \rceil \right\}$$

$$\stackrel{7}{\Rightarrow} \left\{ \widetilde{w}_2(\ell) = 0, \quad \forall \ell \geq \lceil \frac{k_0}{P} \rceil + \widetilde{\delta} \right\}$$

$$\Leftrightarrow \left\{ \begin{bmatrix} w_2(P\ell) \\ \vdots \\ w_2(P\ell + (P-1)) \end{bmatrix} = 0, \quad \forall \ell \geq \lceil \frac{k_0}{P} \rceil + \widetilde{\delta} \right\}$$

$$\Rightarrow \left\{ w_2(k) = 0, \quad \forall k \geq k_0 + P\widetilde{\delta} \right\}.$$

This shows that w_2 is δ*-reconstructible* from w_1 with $\delta = P\widetilde{\delta}$.

$\square$

The behavioral reconstructibility characterization of time-invariant systems, given in Subsection 2.5.1, see Theorem 2.5.4, together with Theorem 5.1.1, allows us to conclude the following.

Proposition 5.1.2 *Let* $\Sigma = (\mathbb{Z}, \mathbb{R}^{q_1+q_2}, \mathfrak{B})$ *be a* P*-periodic system described by*

$$\mathfrak{B} := \left\{ (w_1, w_2) \in \left(\mathbb{R}^{q_1+q_2}\right)^{\mathbb{Z}} \mid \left(R_2\left(\sigma, \sigma^{-1}\right) w_2\right)(Pk) = \left(R_1\left(\sigma, \sigma^{-1}\right) w_1\right)(Pk),\ k \in \mathbb{Z} \right\},$$

with $R_i(\xi, \xi^{-1}) \in \mathbb{R}^{g \times q_i}[\xi, \xi^{-1}]$, $i = 1, 2$. *Then* w_2 *is reconstructible from* w_1 *if and only if*

$$\operatorname{rank} R_2^L\left(\lambda, \lambda^{-1}\right) = Pq_2, \quad \forall \lambda \in \mathbb{C} \setminus \{0\},$$

or, equivalently, if and only if $R_2^L(\xi, \xi^{-1})$ *is a right-prime matrix over* $\mathbb{R}[\xi, \xi^{-1}]$. $\diamond$

[7] By the $\widetilde{\delta}$*-reconstructibility* of Lw_2 from Lw_1.

Taking into account that the right-primeness of a matrix corresponds to the left-primeness of its transposed, and minding Lemma 4.1.4, this result can be formulated as follows in terms of the original matrix R_2.

Proposition 5.1.3 *With the notation of* Proposition 5.1.2*,* w_2 *is reconstructible from* w_1 *if and only if the matrix* $R_2\left(\xi, \xi^{-1}\right)$ *is* P*-right-prime.* $\diamond$

From here on, whenever in a dynamical system, w_2 is reconstructible from w_1, we simply say that $\mathfrak{B}$ is reconstructible with respect to w_2.

§5.2 Reconstructibility of periodic state space systems

Consider, once more as already done in Section 4.2, the state space description (3.16), that is, the state space description

$$\begin{cases} (\sigma x)(k) = A(k)\,x(k) + B(k)\,u(k) \\ \quad y(k) = C(k)\,x(k) + D(k)\,u(k) \end{cases} \qquad k \in \mathbb{Z}. \tag{5.2}$$

Defining $w := \begin{bmatrix} u^T & y^T \end{bmatrix}^T$ and $v := x$, we obtain the following latent variable representation

$$\left(R_t\left(\sigma, \sigma^{-1}\right) w\right)(Pk+t) = \left(M_t\left(\sigma, \sigma^{-1}\right) v\right)(Pk+t), \quad t = 0, \ldots, P-1,\ k \in \mathbb{Z},$$

with

$$R_t\left(\xi, \xi^{-1}\right) = \begin{bmatrix} B(t) & 0 \\ -D(t) & I_p \end{bmatrix} \quad \text{and} \quad M_t\left(\xi, \xi^{-1}\right) = \begin{bmatrix} \xi I_n - A(t) \\ C(t) \end{bmatrix},$$

or still the P-PLVR

$$\left(R\left(\sigma, \sigma^{-1}\right) w\right)(Pk) = \left(M\left(\sigma, \sigma^{-1}\right) v\right)(Pk), \qquad k \in \mathbb{Z}, \tag{5.3}$$

with $R\left(\xi, \xi^{-1}\right)$ and $M\left(\xi, \xi^{-1}\right)$ given by

$$\left[\begin{array}{cc} B(0) & 0 \\ -D(0) & I_p \\ \hline \xi B(1) & 0 \\ -\xi D(1) & \xi I_p \\ \hline \vdots & \vdots \\ \hline \xi^{P-1}B(P-1) & 0 \\ -\xi^{P-1}D(P-1) & \xi^{P-1}I_p \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} \xi I_n - A(0) \\ C(0) \\ \hline \xi(\xi I_n - A(1)) \\ \xi C(1) \\ \hline \vdots \\ \hline \xi^{P-1}(\xi I_n - A(P-1)) \\ \xi^{P-1}C(P-1) \end{array}\right],$$

respectively.

Consequently, if $\mathfrak{B}$ is the behavior formed by the (w,v)-trajectories that satisfy (5.3), the corresponding lifted behavior $L\mathfrak{B}$ is described by

$$\left(R^L\left(\sigma,\sigma^{-1}\right)(Lw)\right)(k) = \left(M^L\left(\sigma,\sigma^{-1}\right)(Lv)\right)(k), \qquad k \in \mathbb{Z},$$

where $M^L(\xi,\xi^{-1}) \in \mathbb{R}^{(n+p)P\times nP}[\xi,\xi^{-1}]$ is equal to

$$\left[\begin{array}{c|c|c|c} -A(0) & I_n & \cdots & 0 \\ C(0) & 0 & \cdots & 0 \\ 0 & -A(1) & \cdots & 0 \\ 0 & C(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi I_n & 0 & \cdots & -A(P-1) \\ 0 & 0 & \cdots & C(P-1) \end{array}\right].$$

Taking Proposition 5.1.2 into account we conclude that $\mathfrak{B}$ is reconstructible with respect to x if and only if

$$\operatorname{rank} M^L\left(\lambda,\lambda^{-1}\right) = nP, \qquad \forall\lambda \in \mathbb{C}\backslash\{0\}.$$

For the sake of simplicity, we now consider that $P = 2$, but our reasonings also apply to the general case. We then have

$$M^L\left(\xi,\xi^{-1}\right) = \left[\begin{array}{c|c} -A(0) & I_n \\ C(0) & 0 \\ \xi I_n & -A(1) \\ 0 & C(1) \end{array}\right].$$

By performing the block-column operation $C_1 \leftarrow C_1 + C_2 A(0)$, where C_j is the j^{th} block-column of M^L, we obtain the following matrix

$$\left[\begin{array}{c|c} 0 & I_n \\ \hline C(0) & 0 \\ \xi I_n - A(1)A(0) & -A(1) \\ C(1)A(0) & C(1) \end{array}\right]. \tag{5.4}$$

Clearly the rank of the original M^L matrix coincides with the rank of matrix (5.4) and, therefore,

$$\forall \lambda \in \mathbb{C}, \quad \operatorname{rank} M^L\left(\lambda, \lambda^{-1}\right) = n + \operatorname{rank}\begin{bmatrix} \lambda I_n - A(1)A(0) \\ C(0) \\ C(1)A(0) \end{bmatrix} = n + \operatorname{rank}\begin{bmatrix} \lambda I_n - A_0 \\ C_0 \end{bmatrix},$$

with $A_0,\ C_0$ as in (1.2), (1.4), respectively, that is,

$$A_0 = A(1)A(0)$$
$$C_0 = \begin{bmatrix} C(0) \\ C(1)A(0) \end{bmatrix}.$$

Therefore $\mathfrak{B}$ is behaviorally reconstructible with respect to x if and only if

$$\operatorname{rank}\begin{bmatrix} \lambda I_n - A_0 \\ C_0 \end{bmatrix} = n, \qquad \forall \lambda \in \mathbb{C}\backslash\{0\}.$$

Suppose now that for some $\lambda^* \in \mathbb{C}\backslash\{0\}$, $\operatorname{rank}\begin{bmatrix} \lambda^* I_n - A_0 \\ C_0 \end{bmatrix} < n$. This means that there exists $0 \neq v^* \in \mathbb{R}^{n\times 1}$ such that

$$\begin{bmatrix} \lambda^* I_n - A_0 \\ C_0 \end{bmatrix} v^* = 0,$$

i.e.,

$$\begin{bmatrix} \lambda^* I_n - A(1)A(0) \\ C(0) \\ C(1)A(0) \end{bmatrix} v^* = 0.$$

This is equivalent to

$$\left(\lambda^* I_n - A(1)A(0)\right) v^* = 0;\text{[8]} \tag{5.5}$$

$$C(0)v^* = 0; \tag{5.6}$$

$$C(1)A(0)v^* = 0. \tag{5.7}$$

[8] v^* is an eigenvector of $A(1)A(0)$ associated to the eigenvalue λ^*.

Consequently, the product

$$\begin{bmatrix} \lambda^* I_n - A_1 \\ C_1 \end{bmatrix} A(0)\, v^*,$$

where

$$A_1 = A(0)\, A(1)$$

$$C_1 = \begin{bmatrix} C(1) \\ C(0)\, A(1) \end{bmatrix},$$

is given by:

$$\begin{bmatrix} \lambda^* I_n - A(0)\, A(1) \\ C(1) \\ C(0)\, A(1) \end{bmatrix} A(0)\, v^* = \begin{bmatrix} A(0)\, \lambda^* v^* - A(0)\, A(1)\, A(0)\, v^* \\ \underbrace{C(1)\, A(0)\, v^*}_{=0,\text{ by (5.7)}} \\ C(0)\, \underbrace{A(1)\, A(0)\, v^*}_{=\lambda^* v^*,\text{ by (5.5)}} \end{bmatrix}$$

$$= \begin{bmatrix} A(0)\, \underbrace{\left(\lambda^* v^* - A(1)\, A(0)\, v^*\right)}_{=0,\text{ by (5.5)}} \\ 0 \\ \lambda^* \underbrace{C(0)\, v^*}_{=0,\text{ by (5.6)}} \end{bmatrix} = 0.$$

Since $A(0)\, v^* \neq 0$ (otherwise v^* would be an eigenvector of $A(0)$ associated to the eigenvalue zero, which is not the case since we have assumed that $\lambda^* \neq 0$), we conclude that also

$$\operatorname{rank} \begin{bmatrix} \lambda^* I_n - A_1 \\ C_1 \end{bmatrix} < n.$$

Taking into account that this procedure can be reversed, this yields that

$$\left\{ \operatorname{rank} \begin{bmatrix} \lambda I_n - A_0 \\ C_0 \end{bmatrix} = n,\ \forall \lambda \in \mathbb{C} \backslash \{0\} \right\} \Leftrightarrow \left\{ \operatorname{rank} \begin{bmatrix} \lambda I_n - A_1 \\ C_1 \end{bmatrix} = n,\ \forall \lambda \in \mathbb{C} \backslash \{0\} \right\}.$$

Noting that this reasoning can be easily extended to the general P-periodic case, we obtain the next result.

Theorem 5.2.1 *Let Σ be a P-periodic state space system, described as in* (5.2)*, and let $\Sigma_t = (A_t, B_t, C_t, D_t)$ be the P time-invariant systems obtained by the invariant dynamical decomposition described in* Section 1.2*. Then the following conditions are equivalent:*

i) *The behavior $\mathfrak{B}$ of Σ is behaviorally reconstructible with respect to x;*

ii) $\operatorname{rank}\begin{bmatrix} \lambda I_n - A_t \\ C_t \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \setminus \{0\}$, *for at least one t in $\{0, \ldots, P-1\}$;*

iii) $\operatorname{rank}\begin{bmatrix} \lambda I_n - A_t \\ C_t \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \setminus \{0\}$, *for all t in $\{0, \ldots, P-1\}$.*

$\diamond$

Combining Theorems 1.4.3, 1.4.5 and 5.2.1, we immediately conclude that:

Theorem 5.2.2 *The behavior $\mathfrak{B}$ of a P-periodic state space system Σ is (behaviorally) reconstructible with respect to x if and only if Σ is completely state reconstructible.* $\diamond$

§5.3 Observability

In Section 2.5, two definitions of observability have been considered, namely: *forward-observability*; *Willems-observability*. Concerning the property of Willems-observability, in Proposition 2.5.5 a one-to-one connection with the property of behavioral reconstructibility was established, for the time-invariant case. Thus, by using Theorem 5.1.1, we may state that a P-periodic behavior $\mathfrak{B}$, with variable $w = (w_1, w_2)$, is reconstructible (with respect to w_2) if and only if, in the associated lifted behavior $L\mathfrak{B}$, Lw_2 is Willems-observable from Lw_1. Therefore, by proving that Willems-observability of behavior $\mathfrak{B}$ is equivalent to the Willems-observability of its associated lifted behavior $L\mathfrak{B}$, we may conclude that Willems-observability and reconstructibility are equivalent properties also in the periodic case.

Proposition 5.3.1 *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^{q_1+q_2}, \mathfrak{B})$ be a P-periodic system described by*

$$\mathfrak{B} := \left\{ (w_1, w_2) \in \left(\mathbb{R}^{q_1+q_2}\right)^{\mathbb{Z}} \mid \left(R_2\left(\sigma, \sigma^{-1}\right) w_2\right)(Pk) = \left(R_1\left(\sigma, \sigma^{-1}\right) w_1\right)(Pk),\ k \in \mathbb{Z} \right\},$$

with $R_i\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{g \times q_i}\left[\xi, \xi^{-1}\right]$, $i = 1, 2$. Then w_2 is Willems-observable from w_1 if and only if, in $L\mathfrak{B}$, Lw_2 is Willems-observable from Lw_1.

Proof.

($\Rightarrow$): Assume that $\mathfrak{B}$ is Willems-observable, i.e., that

$$\{w_1(k) = 0, \quad \forall k \in \mathbb{Z}\} \Rightarrow \{w_2(k) = 0, \quad \forall k \in \mathbb{Z}\}.$$

Let $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ be such that $Lw_i = \widetilde{w}_i,\ i = 1, 2$. Then,

$$\begin{aligned}
\{\widetilde{w}_1(k) = 0, \quad \forall k \in \mathbb{Z}\} &\Leftrightarrow \left\{ \begin{bmatrix} w_1(Pk) \\ \vdots \\ w_1(Pk + (P-1)) \end{bmatrix} = 0, \quad \forall k \in \mathbb{Z} \right\} \\
&\Leftrightarrow \{w_1(k) = 0, \quad \forall k \in \mathbb{Z}\} \\
&\Rightarrow \{w_2(k) = 0, \quad \forall k \in \mathbb{Z}\} \\
&\Leftrightarrow \left\{ \begin{bmatrix} w_2(Pk) \\ \vdots \\ w_2(Pk + (P-1)) \end{bmatrix} = 0, \quad \forall k \in \mathbb{Z} \right\} \\
&\Leftrightarrow \{\widetilde{w}_2(k) = 0, \quad \forall k \in \mathbb{Z}\},
\end{aligned}$$

showing that $L\mathfrak{B}$ is Willems-observable;

($\Leftarrow$): Assume now that $L\mathfrak{B}$ is Willems-observable. Let $w_1, w_2 \in \mathfrak{B}$ and define $\widetilde{w}_i = Lw_i,\ i = 1, 2$. Then,

$$\begin{aligned}
\{w_1(k) = 0, \quad \forall k \in \mathbb{Z}\} &\Leftrightarrow \left\{ \begin{bmatrix} w_1(Pk) \\ \vdots \\ w_1(Pk + (P-1)) \end{bmatrix} = 0, \quad \forall k \in \mathbb{Z} \right\} \\
&\Leftrightarrow \{\widetilde{w}_1(k) = 0, \quad \forall k \in \mathbb{Z}\} \\
&\Rightarrow \{\widetilde{w}_2(k) = 0, \quad \forall k \in \mathbb{Z}\} \\
&\Leftrightarrow \left\{ \begin{bmatrix} w_2(Pk) \\ \vdots \\ w_2(Pk + (P-1)) \end{bmatrix} = 0, \quad \forall k \in \mathbb{Z} \right\} \\
&\Leftrightarrow \{w_2(k) = 0, \quad \forall k \in \mathbb{Z}\},
\end{aligned}$$

showing that $\mathfrak{B}$ is Willems-observable.

$\square$

Corollary 5.3.2 *With the notation of* Proposition 5.3.1, *w_2 is reconstructible from w_1 if and only if it is Willems-observable from w_1.* ◇

As for the property of forward-observability, the situation is somewhat different, since there is no total correspondence between what happens for a periodic behavior and for its associated lifted system.

Theorem 5.3.3 *Let $\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B})$ be a P-periodic system whose system variable w is partitioned as $w = (w_1, w_2)$. Then w_2 is* forward-observable *from w_1 only if in the associated lifted system, $\Sigma^L = \left(\mathbb{Z}, \mathbb{R}^{Pq}, L\mathfrak{B}\right)$, Lw_2 is* forward-observable *from Lw_1.*

Proof. Assume that w_2 is forward-observable from w_1. Let $\widetilde{w}_1, \widetilde{w}_2 \in L\mathfrak{B}$ and $w_1, w_2 \in \mathfrak{B}$ be such that $Lw_i = \widetilde{w}_i,\ i = 1, 2$. Then,

$$\{\widetilde{w}_1(k) = 0, \quad \forall k \geq k_0\} \Leftrightarrow \left\{ \begin{bmatrix} w_1(Pk) \\ \vdots \\ w_1(Pk + (P-1)) \end{bmatrix} = 0, \quad \forall k \geq k_0 \right\}$$

$$\Leftrightarrow \{w_1(\ell) = 0, \quad \forall \ell \geq Pk_0\}$$

$$\overset{9}{\Rightarrow} \{w_2(\ell) = 0, \quad \forall \ell \geq Pk_0\}$$

$$\Leftrightarrow \left\{ \begin{bmatrix} w_2(Pk) \\ \vdots \\ w_2(Pk + (P-1)) \end{bmatrix} = 0, \quad \forall k \geq k_0 \right\}$$

$$\Leftrightarrow \{\widetilde{w}_2(k) = 0, \quad \forall k \geq k_0\},$$

showing that Lw_2 is forward-observable from Lw_1. □

The next example shows that the forward-observability of the lifted system does not imply the forward-observability of the original periodic system.

Example 5.3.4 *Consider the 2-periodic system $\Sigma = (\mathbb{Z}, \mathbb{R}^2, \mathfrak{B})$, with behavior $\mathfrak{B}$ described by*

$$\mathfrak{B} \ \sim \ w_2(2k) = w_2(2k+1) = w_1(2k), \quad k \in \mathbb{Z}.$$

[9] By the assumption that w_2 is forward-observable from w_1.

The associated lifted behavior is described by

$$L\mathfrak{B} \;\sim\; (Lw_2)(k) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (Lw_1) \right)(k), \quad k \in \mathbb{Z}.$$

Thus Lw_2 is, obviously, forward-observable from Lw_1. However, the same does not holds for the periodic behavior $\mathfrak{B}$. In order to verify this, take a trajectory (w_1, w_2) such that

$$(w_1, w_2)(k) = \begin{cases} (1,1) & \textit{if } k \in (-\infty, 0], \\ (0,1) & \textit{if } k = 1, \\ (0,0) & \textit{if } k \in [2, +\infty). \end{cases}$$

This trajectory belongs to the behavior and verifies

$$w_1(k) = 0, \quad k \geq 1 \quad \text{but} \quad w_2(1) \neq 0,$$

showing that w_2 is not forward-observable from w_1. ◇

§5.4 Conclusion

The notions of behavioral reconstructibility and forward-observability have been introduced now within the P-periodic case. We proved that in a P-periodic system the variable w_2 is reconstructible from w_1 if and only if, in its associated lifted system, the correspondent variable Lw_2 is reconstructible from Lw_1. However, this does not happen with forward-observability. Indeed, the forward-observability of Lw_2 from Lw_1 in the lifted system is a necessary but not sufficient condition for the forward-observability of w_2 from w_1 in the original periodic system. In the case of reconstructibility, the stated equivalence allowed us to obtain a characterization by means of a rank condition on R_2^L or directly in R_2 by means of a P-right-primeness condition. Furthermore, we proved that also in the case of a P-periodic behavior, Willems-observability is equivalent to (behavioral) reconstructibility. Finally, we applied these results to the particular case of P-periodic state space systems, and concluded that behavioral reconstructibility is in fact equivalent to the notion of complete state reconstructibility as introduced in Chapter 2.

"Pasmo sempre quando acabo qualquer coisa. Pasmo e desolo-me. O meu instinto de perfeição deveria inibir-me de acabar; deveria inibir-me até de dar começo. Mas distraio-me e faço. O que consigo é um produto, em mim, não de uma aplicação da vontade, mas de uma cedência dela. Começo porque não tenho força para pensar; acabo porque não tenho alma para suspender."

— Fernando Pessoa

Conclusions

In this thesis we continued the work carried out by Margreet Kuijper and Jan C. Willems, in [47], in order to give an answer to some basic questions that arise within the behavioral approach to periodic systems. In this context, we have analysed some properties of kernel-type representations for periodic systems, such as equivalence and minimality. Furthermore, we have introduced and analysed latent variable-type and, in particular, image-type representations, which generalize their time-invariant counterparts. We have also proved the existence of a variable elimination procedure, somehow analogous to the time-invariant case.

At the level of system theoretic properties, we have investigated the controllability and autonomy of a periodic system, based on the characterization of such properties for the associated lifted system, and have shown the existence of an autonomous/controllable decomposition of periodic behaviors similarly to the time-invariant one. Motivated by the time-invariant case, where controllability and autonomy are strictly related to the presence or absence of free variables, we have analysed this issue for the periodic case and have obtained a new concept of variable freeness which is suitable both for the periodic and the time-invariant cases. In relation to our notion of freeness, we defined the concept of P-periodic input, as well as input-output structures in P-periodic systems.

Finally, once more at the level of system theoretic properties, we investigated the reconstructibility and the forward-observability of a periodic system. We have proved that a periodic system and its associated lifted system behave in consonance with respect to reconstructibility, whereas the same does not happen with forward-observability. Furthermore, we proved that Willems-observability and reconstructibility turn out to be equivalent also for periodic behaviors.

Throughout this thesis we have considered the particular case of periodic state space systems in order to give more insight into the relationship between the classical and the behavioral approaches. In this context, we proved the equivalence between the behavioral properties of controllability and reconstructibility and their classic counterparts, i.e., complete state controllability and complete state reconstructibility, respectively.

In our opinion the obtained results may constitute a starting point for other challenges such as the study of control problems for periodic systems within the behavioral framework.

Appendix

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

— Bertrand Russell

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

— Albert Einstein

A
The lifting system

§A.1 The lifted map

Note that the *lifting* map, introduced in Section 3.1, given by

$$\begin{array}{rccccc} L: & (\mathbb{R}^q)^{\mathbb{Z}} & \longrightarrow & \left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}} & & \\ & w & \mapsto & Lw: & \mathbb{Z} \longrightarrow & \mathbb{R}^{Pq} \\ & & & & k \mapsto & \begin{bmatrix} w(Pk) \\ \vdots \\ w(Pk+P-1) \end{bmatrix}, \end{array}$$

is in fact linear, since:

$$
\begin{aligned}
\left(L\left(\alpha w_1+\beta w_2\right)\right)(k) &:= \begin{bmatrix} \left(\alpha w_1+\beta w_2\right)(Pk) \\ \vdots \\ \left(\alpha w_1+\beta w_2\right)(Pk+P-1) \end{bmatrix} \\
&= \begin{bmatrix} \left(\alpha w_1\right)(Pk)+\left(\beta w_2\right)(Pk) \\ \vdots \\ \left(\alpha w_1\right)(Pk+P-1)+\left(\beta w_2\right)(Pk+P-1) \end{bmatrix} \\
&= \alpha \begin{bmatrix} w_1(Pk) \\ \vdots \\ w_1(Pk+P-1) \end{bmatrix} + \beta \begin{bmatrix} w_2(Pk) \\ \vdots \\ w_2(Pk+P-1) \end{bmatrix} \\
&= \alpha\left(Lw_1\right)(k)+\beta\left(Lw_2\right)(k), \quad \alpha,\beta\in\mathbb{R},\ w_{1,2}(\cdot)\in\left(\mathbb{R}^q\right)^{\mathbb{Z}}.
\end{aligned}
$$

Note also that

$$
L\sigma = \begin{bmatrix} 0 & I_q & 0 & \cdots \\ \vdots & & \ddots & \\ & & & I_q \\ \sigma I_q & 0 & \cdots & 0 \end{bmatrix} L.
$$

In fact, letting $(Lw) =: \widetilde{w}$, with $\widetilde{w}_{t+1}(k) := w(Pk+t)$, $t = 0,\ldots,P-1$, we get

$$
\begin{aligned}
\left(\begin{bmatrix} 0 & I_q & 0 & \cdots \\ \vdots & & \ddots & \\ & & & I_q \\ \sigma I_q & 0 & \cdots & 0 \end{bmatrix}(Lw)\right)(k) &= \left(\begin{bmatrix} 0 & I_q & 0 & \cdots \\ \vdots & & \ddots & \\ & & & I_q \\ \sigma I_q & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} \widetilde{w}_1 \\ \vdots \\ \widetilde{w}_P \end{bmatrix}\right)(k) \\
&= \begin{bmatrix} \widetilde{w}_2 \\ \vdots \\ \widetilde{w}_P \\ \sigma\widetilde{w}_1 \end{bmatrix}(k) = \begin{bmatrix} \widetilde{w}_2(k) \\ \vdots \\ \widetilde{w}_P(k) \\ \widetilde{w}_1(k+1) \end{bmatrix} \\
&= \begin{bmatrix} w(Pk+1) \\ \vdots \\ w(Pk+P-1) \\ w(P(k+1)) \end{bmatrix} = \begin{bmatrix} w(Pk+1) \\ \vdots \\ w(Pk+P-1) \\ w(Pk+P) \end{bmatrix}.
\end{aligned}
$$

On the other hand,

$$((L\sigma)w)(k) = (L(\sigma w))(k) := \begin{bmatrix} (\sigma w)(Pk) \\ \vdots \\ (\sigma w)(Pk+P-1) \end{bmatrix} = \begin{bmatrix} w(Pk+1) \\ \vdots \\ w(Pk+P) \end{bmatrix}.$$

Observe now that $L\sigma^P = \sigma L$ since

$$\begin{aligned} \left(L\left(\sigma^P w\right)\right)(k) &= \begin{bmatrix} \left(\sigma^P w\right)(Pk) \\ \vdots \\ \left(\sigma^P w\right)(Pk+P-1) \end{bmatrix} = \begin{bmatrix} w(Pk+P) \\ \vdots \\ w(Pk+P+P-1) \end{bmatrix} \\ &= \begin{bmatrix} w(P(k+1)) \\ \vdots \\ w(P(k+1)+P-1) \end{bmatrix} = (Lw)(k+1) = (\sigma(Lw))(k). \end{aligned}$$

Consequently Proposition 3.1.4–i) has now an easy and direct proof, due to

$$\sigma L\mathfrak{B} = L\sigma^P\mathfrak{B} = L\mathfrak{B}.$$

§A.2 The kernel representation matrix R^L

Recall the decomposition of $R(\xi, \xi^{-1})$ given by (3.6–3.8), i.e.,

$$\begin{aligned} R\left(\xi, \xi^{-1}\right) &= R_0^L\left(\xi^P, \xi^{-P}\right) + \xi R_1^L\left(\xi^P, \xi^{-P}\right) + \cdots + \xi^{P-1} R_{P-1}^L\left(\xi^P, \xi^{-P}\right) \\ &= R^L\left(\xi^P, \xi^{-P}\right)\Omega_{P,q}(\xi), \end{aligned}$$

with

$$\Omega_{P,q}(\xi) := \begin{bmatrix} I_q & \xi I_q & \cdots & \xi^{P-1} I_q \end{bmatrix}^T$$

and

$$R^L\left(\xi, \xi^{-1}\right) = \begin{bmatrix} R_0^L(\xi, \xi^{-1}) & R_1^L(\xi, \xi^{-1}) & \cdots & R_{P-1}^L(\xi, \xi^{-1}) \end{bmatrix}.$$

Another easy way of ascertaining the uniqueness of this decomposition is by assuming that $R(\xi, \xi^{-1})$ has two different decompositions, namely:

$$R\left(\xi, \xi^{-1}\right) = R^L\left(\xi^P, \xi^{-P}\right)\Omega_{P,q}(\xi)$$

and

$$R\left(\xi, \xi^{-1}\right) = R'^L\left(\xi^P, \xi^{-P}\right)\Omega_{P,q}(\xi).$$

Then

$$R^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}\left(\xi\right) = R'^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}\left(\xi\right),$$

i.e.,

$$\begin{aligned} &R_0^L\left(\xi^P,\xi^{-P}\right)+\xi R_1^L\left(\xi^P,\xi^{-P}\right)+\cdots+\xi^{P-1}R_{P-1}^L\left(\xi^P,\xi^{-P}\right)=\\ &R_0'^L\left(\xi^P,\xi^{-P}\right)+\xi R_1'^L\left(\xi^P,\xi^{-P}\right)+\cdots+\xi^{P-1}R_{P-1}'^L\left(\xi^P,\xi^{-P}\right). \end{aligned}$$

Or, equivalently,

$$\begin{aligned} &\left[R_0^L\left(\xi^P,\xi^{-P}\right)-R_0'^L\left(\xi^P,\xi^{-P}\right)\right]+\xi\left[R_1^L\left(\xi^P,\xi^{-P}\right)-R_1'^L\left(\xi^P,\xi^{-P}\right)\right]+\\ &\qquad\cdots+\xi^{P-1}\left[R_{P-1}^L\left(\xi^P,\xi^{-P}\right)-R_{P-1}'^L\left(\xi^P,\xi^{-P}\right)\right]=0. \end{aligned} \tag{A.1}$$

Since the terms

$$\xi^t\left[R_t^L\left(\xi^P,\xi^{-P}\right)-R_t'^L\left(\xi^P,\xi^{-P}\right)\right],\quad t=0,\ldots,P-1,$$

have no monomials of the same order, (A.1) is equivalent to:

$$\begin{aligned} R_0^L\left(\xi^P,\xi^{-P}\right)&=R_0'^L\left(\xi^P,\xi^{-P}\right)\\ R_1^L\left(\xi^P,\xi^{-P}\right)&=R_1'^L\left(\xi^P,\xi^{-P}\right)\\ &\vdots\\ R_{P-1}^L\left(\xi^P,\xi^{-P}\right)&=R_{P-1}'^L\left(\xi^P,\xi^{-P}\right), \end{aligned}$$

i.e.,

$$R^L\left(\xi^P,\xi^{-P}\right)=R'^L\left(\xi^P,\xi^{-P}\right).$$

In Section 3.2, we have pointed out that, from the decomposition (3.6–3.8) and the definition of the lifted trajectory Lw associated to w, (3.4) can be written as

$$\left(R^L\left(\sigma,\sigma^{-1}\right)\left(Lw\right)\right)\left(k\right)=0,\quad k\in\mathbb{Z}.$$

Furthermore, we have stated Lemma 3.2.3, cited on [47], but without giving any explicit expression for matrix R^L. In fact its explicit form can be found in [47], and we give here an idea of how to obtain it. For that purpose decompose each of the R_t's ($t=0,\ldots,P-1$) in terms of their powers ξ^j, with j taken modulo P, i.e., write

$$R_t\left(\xi,\xi^{-1}\right)=R_t^0\left(\xi^P,\xi^{-P}\right)+\xi^{-1}R_t^1\left(\xi^P,\xi^{-P}\right)+\cdots+\xi^{-(P-1)}R_t^{P-1}\left(\xi^P,\xi^{-P}\right). \tag{A.2}$$

Observe that this is no more than our decomposition (3.6–3.8), (block) row-by-row.

Consequently, the first equation in (3.3) - that is for $t = 0$ - will be equivalent to

$$\left(\left(R_0^0\left(\sigma^P,\sigma^{-P}\right)+\sigma^{-1}R_0^1\left(\sigma^P,\sigma^{-P}\right)+\cdots+\sigma^{-(P-1)}R_0^{P-1}\left(\sigma^P,\sigma^{-P}\right)\right)w\right)(Pk)=0$$

$$\begin{gathered}\left(R_0^0\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk)+\left(\sigma^{-1}R_0^1\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk)+\cdots\\ \cdots+\left(\sigma^{-(P-1)}R_0^{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk)=0\end{gathered}$$

$$\begin{gathered}\left(R_0^0\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk)+\left(R_0^1\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk-1)+\cdots\\ \cdots+\left(R_0^{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk-P+1)=0\end{gathered}$$

$$\begin{aligned}\left(R_0^0\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk)+\left(R_0^1\left(\sigma^P,\sigma^{-P}\right)w\right)(P(k-1)+P-1)+\cdots\\ \cdots+\left(R_0^{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(P(k-1)+1)=0\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\begin{bmatrix} R_0^0\left(\sigma^P,\sigma^{-P}\right) & R_0^{P-1}\left(\sigma^P,\sigma^{-P}\right) & \cdots & R_0^1\left(\sigma^P,\sigma^{-P}\right)\end{bmatrix}\\ &\qquad\qquad\qquad\times\begin{bmatrix} w(Pk)\\ w(P(k-1)+1)\\ \vdots\\ w(P(k-1)+P-1)\end{bmatrix}=0,\end{aligned}$$

$$\begin{aligned}&\begin{bmatrix} R_0^0\left(\sigma^P,\sigma^{-P}\right) & \sigma^{-P}R_0^{P-1}\left(\sigma^P,\sigma^{-P}\right) & \cdots & \sigma^{-P}R_0^1\left(\sigma^P,\sigma^{-P}\right)\end{bmatrix}\\ &\qquad\qquad\qquad\times\underbrace{\begin{bmatrix} w(Pk)\\ w(Pk+1)\\ \vdots\\ w(Pk+P-1)\end{bmatrix}}_{=:\widetilde{w}(k)}=0. \qquad (A.3)\end{aligned}$$

Since for every element $\widetilde{w}_t$ of $\widetilde{w}$ we have

$$(\sigma\widetilde{w}_t)(k)=\widetilde{w}_t(k+1)=w(P(k+1)+t)=\left(\sigma^P w\right)(Pk+t),\quad t=0,\ldots,P-1,$$

we may rewrite equality (A.3) as

$$\left(\begin{bmatrix} R_0^0\left(\sigma,\sigma^{-1}\right) & \sigma^{-1}R_0^{P-1}\left(\sigma,\sigma^{-1}\right) & \cdots & \sigma^{-1}R_0^1\left(\sigma,\sigma^{-1}\right)\end{bmatrix}\widetilde{w}\right)(k)=0.$$

Analogously, the second equation in (3.3) - that is for $t=1$ - will be equivalent to

$$\begin{aligned}&\Big(\big(R_1^0\left(\sigma^P,\sigma^{-P}\right)+\sigma^{-1}R_1^1\left(\sigma^P,\sigma^{-P}\right)+\sigma^{-2}R_1^2\left(\sigma^P,\sigma^{-P}\right)+\cdots\\&\qquad\cdots+\sigma^{-(P-1)}R_1^{P-1}\left(\sigma^P,\sigma^{-P}\right)\big)\,w\Big)\left(Pk+1\right)=0\end{aligned}$$

$$\begin{gathered}\left(R_1^0\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk+1\right)+\left(\sigma^{-1}R_1^1\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk+1\right)\\+\left(\sigma^{-2}R_1^2\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk+1\right)+\cdots+\left(\sigma^{-(P-1)}R_1^{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk+1\right)=0\end{gathered}$$

$$\begin{gathered}\left(R_1^0\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk+1\right)+\left(R_1^1\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk\right)\\+\left(R_1^2\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk-1\right)+\cdots+\left(R_1^{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk-P+2\right)=0\end{gathered}$$

$$\begin{gathered}\left(R_1^0\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk+1\right)+\left(R_1^1\left(\sigma^P,\sigma^{-P}\right)w\right)\left(Pk\right)\\+\left(R_1^2\left(\sigma^P,\sigma^{-P}\right)w\right)\left(P\left(k-1\right)+P-1\right)+\cdots+\left(R_1^{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)\left(P\left(k-1\right)+2\right)=0\end{gathered}$$

or, equivalently,

$$\begin{aligned}&\begin{bmatrix}R_1^1\left(\sigma^P,\sigma^{-P}\right) & R_1^0\left(\sigma^P,\sigma^{-P}\right) & R_1^{P-1}\left(\sigma^P,\sigma^{-P}\right) & \cdots & R_1^2\left(\sigma^P,\sigma^{-P}\right)\end{bmatrix}\\&\qquad\times\begin{bmatrix}w\left(Pk\right)\\ w\left(Pk+1\right)\\ w\left(P\left(k-1\right)+2\right)\\ \vdots\\ w\left(P\left(k-1\right)+P-1\right)\end{bmatrix}=0\end{aligned}$$

$$\begin{aligned}&\begin{bmatrix}R_1^1\left(\sigma^P,\sigma^{-P}\right) & R_1^0\left(\sigma^P,\sigma^{-P}\right) & \sigma^{-P}R_1^{P-1}\left(\sigma^P,\sigma^{-P}\right) & \cdots & \sigma^{-P}R_1^2\left(\sigma^P,\sigma^{-P}\right)\end{bmatrix}\\&\qquad\qquad\times\underbrace{\begin{bmatrix}w\left(Pk\right)\\ w\left(Pk+1\right)\\ w\left(Pk+2\right)\\ \vdots\\ w\left(Pk+P-1\right)\end{bmatrix}}_{=:\widetilde{w}(k)}=0.\end{aligned}$$

and therefore

$$\left(\begin{bmatrix}R_1^1\left(\sigma,\sigma^{-1}\right) & R_1^0\left(\sigma,\sigma^{-1}\right) & \sigma^{-1}R_1^{P-1}\left(\sigma,\sigma^{-1}\right) & \cdots & \sigma^{-1}R_1^2\left(\sigma,\sigma^{-1}\right)\end{bmatrix}\widetilde{w}\right)(k)=0.$$

Finally, the last equation in (3.3) - that is for $t = P-1$ - will be equivalent to

$$\begin{aligned}
&\left(\left(R^0_{P-1}\left(\sigma^P,\sigma^{-P}\right)+\sigma^{-1}R^1_{P-1}\left(\sigma^P,\sigma^{-P}\right)+\cdots\right.\right.\\
&\qquad\left.\left.\cdots+\sigma^{-(P-1)}R^{P-1}_{P-1}\left(\sigma^P,\sigma^{-P}\right)\right)w\right)(Pk+P-1)=0\\
&\left(R^0_{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk+P-1)+\left(\sigma^{-1}R^1_{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk+P-1)+\cdots\\
&\qquad\cdots+\left(\sigma^{-(P-1)}R^{P-1}_{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk+P-1)=0\\
&\left(R^0_{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk+P-1)+\left(R^1_{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk+P-2)+\cdots\\
&\qquad\cdots+\left(R^{P-1}_{P-1}\left(\sigma^P,\sigma^{-P}\right)w\right)(Pk)=0
\end{aligned}$$

or, equivalently,

$$\begin{bmatrix} R^{P-1}_{P-1}\left(\sigma^P,\sigma^{-P}\right) & \cdots & R^1_{P-1}\left(\sigma^P,\sigma^{-P}\right) & R^0_{P-1}\left(\sigma^P,\sigma^{-P}\right)\end{bmatrix}$$
$$\times\underbrace{\begin{bmatrix} w(Pk)\\ \vdots \\ w(Pk+P-2)\\ w(Pk+P-1)\end{bmatrix}}_{=:\widetilde{w}(k)}=0$$

and therefore

$$\left(\begin{bmatrix} R^{P-1}_{P-1}\left(\sigma,\sigma^{-1}\right) & \cdots & R^1_{P-1}\left(\sigma,\sigma^{-1}\right) & R^0_{P-1}\left(\sigma,\sigma^{-1}\right)\end{bmatrix}\widetilde{w}\right)(k)=0.$$

Consequently we have that $R^L\left(\xi,\xi^{-1}\right)\in\mathbb{R}^{g\times Pq}\left[\xi,\xi^{-1}\right]$ is given by

$$R^L\left(\xi,\xi^{-1}\right):=\begin{bmatrix} R^0_0\left(\xi,\xi^{-1}\right) & \xi^{-1}R^{P-1}_0\left(\xi,\xi^{-1}\right) & \cdots & \xi^{-1}R^1_0\left(\xi,\xi^{-1}\right)\\ R^1_1\left(\xi,\xi^{-1}\right) & R^0_1\left(\xi,\xi^{-1}\right) & \cdots & \xi^{-1}R^2_1\left(\xi,\xi^{-1}\right)\\ \vdots & \vdots & \ddots & \vdots\\ R^{P-1}_{P-1}\left(\xi,\xi^{-1}\right) & R^{P-2}_{P-1}\left(\xi,\xi^{-1}\right) & \cdots & R^0_{P-1}\left(\xi,\xi^{-1}\right)\end{bmatrix},$$

with $g=\sum_{t=0}^{P-1}g_t$.

“That which we persist in doing becomes easier, not that the task itself has become easier, but that our ability to perform it has improved.”
— Ralph Waldo Emerson

“Have no fear of perfection - you’ll never reach it. ”
— Salvador Dali

B
Autonomy characterization

§B.1 Some considerations

Recall Remark 4.4.8. As pointed out, and due to decomposition (3.6–3.8), the following results hold.

$$\begin{aligned} R^L\left(\xi,\xi^{-1}\right) \text{ has fcr (over } \mathbb{R}\left[\xi,\xi^{-1}\right]) &\overset{①}{\Leftrightarrow}_{②} R^L\left(\xi^P,\xi^{-P}\right) \text{ has fcr} \\ &\Rightarrow_{③} R\left(\xi,\xi^{-1}\right) \text{ has fcr} \\ &\not\Leftarrow_{④} R\left(\xi,\xi^{-1}\right) \text{ has fcr} \end{aligned}$$

Clearly step ④ in this scheme has a straight and obvious proof, see Example 4.4.2. Consider now the remaining steps.

① Suppose that $R^L\left(\xi,\xi^{-1}\right)$ is not full column rank. Then

$$\exists\, 0 \neq X\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{Pq\times 1}\left[\xi,\xi^{-1}\right] \text{ s.t. } R^L\left(\xi,\xi^{-1}\right) X\left(\xi,\xi^{-1}\right) = 0$$

which is equivalent to say that

$$\exists X\left(\xi^P,\xi^{-P}\right) \neq 0 \text{ s.t. } R^L\left(\xi^P,\xi^{-P}\right) X\left(\xi^P,\xi^{-P}\right) = 0,$$

which clearly implies that $R^L\left(\xi^P,\xi^{-P}\right)$ is not full column rank;

② Suppose that $R^L\left(\xi^P, \xi^{-P}\right)$ is not full column rank. Then

$$\exists\, 0 \neq X\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{Pq\times 1}\left[\xi, \xi^{-1}\right] \text{ s.t. } R^L\left(\xi^P, \xi^{-P}\right) X\left(\xi, \xi^{-1}\right) = 0$$

which is successively equivalent to, due to the uniqueness of the decomposition,

$$\begin{aligned} R^L\left(\xi^P, \xi^{-P}\right) \underbrace{X^L\left(\xi^P, \xi^{-P}\right)}_{Pq\times P} \Omega_{P,1}(\xi) &= 0 \\ R^L\left(\xi^P, \xi^{-P}\right) X^L\left(\xi^P, \xi^{-P}\right) &= 0 \\ R^L\left(\xi, \xi^{-1}\right) X^L\left(\xi, \xi^{-1}\right) &= 0, \text{ with } X^L\left(\xi, \xi^{-1}\right) \neq 0, \end{aligned}$$

that is $R^L(\xi, \xi^{-1})$ is not full column rank;

③ Suppose that $R(\xi, \xi^{-1})$ is not full column rank. Then

$$\exists\, 0 \neq X\left(\xi, \xi^{-1}\right) \in \mathbb{R}^{q\times 1}\left[\xi, \xi^{-1}\right] \text{ s.t. } R\left(\xi, \xi^{-1}\right) X\left(\xi, \xi^{-1}\right) = 0$$

which is equivalent to

$$R^L\left(\xi^P, \xi^{-P}\right) \Omega_{P,q}(\xi)\, X\left(\xi, \xi^{-1}\right) = 0.$$

Therefore

$$\exists Y\left(\xi, \xi^{-1}\right) := \Omega_{P,q}(\xi)\, X\left(\xi, \xi^{-1}\right) \neq 0 \text{ s.t. } R^L\left(\xi^P, \xi^{-P}\right) Y\left(\xi, \xi^{-1}\right) = 0$$

and thus $R^L\left(\xi^P, \xi^{-P}\right)$ is not full column rank.

Notation

Symbol	**Short description**	**Page**
$\diamond$	end of comment section	
$\square$	end of proof	
$:=$	equal by definition	
$\equiv$	identically equal	
I_n	identity matrix of order n	
$\operatorname{diag}[A_1\ A_2 \cdots A_n]$	$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{bmatrix}$	
det	determinant	82
rank	the rank	
im	image of linear map	
ker	kernel of linear map	
$\mathbb{R}_+$	set of non-negative real numbers	
$\mathbb{Z}_+$	set of non-negative integers	
$\mathbb{R}^{n_1 \times n_2}$	set of real $n_1 \times n_2$ matrices	
$\mathbb{N}$	set of positive integers	
$\mathbb{Z}$	set of integers	
$\mathbb{R}$	set of real numbers	
$\mathbb{C}$	set of complex numbers	
$\mathbb{T}$	$\mathbb{T} \subseteq \mathbb{R}$, time set	
$\mathbb{W}$	signal space	
$\mathbb{W}^{\mathbb{T}}$	set of functions $\mathbb{T} \to \mathbb{W}$	17
$\mathfrak{B}$	a behavior	17
$\mathfrak{B}^{\mathrm{c}}$	the controllable part of behavior $\mathfrak{B}$	25
$\mathfrak{B}^{\mathrm{a}}$	the autonomous part of behavior $\mathfrak{B}$	25
Σ	dynamical system	17
σ, σ^{τ}	the shift-operator	18
L_P	the *lifting* map	44
Σ_P^L / Σ^L	a *lifted* system associated to the system Σ	44
$L_P\mathfrak{B}/L\mathfrak{B}$	a *lifted* behavior associated to the behavior $\mathfrak{B}$	44
$\Omega_{P,q}(\cdot)$	the Ω operator	47

$\mathbb{R}\left[\xi\right]$	the set of real polynomials in the indeterminate ξ	30
$\mathbb{R}\left[\xi,\xi^{-1}\right]$	the set of Laurent-polynomials with real entries	19
$\mathbb{R}^{n_1\times n_2}\left[\xi\right]$	the set of real polynomial $n_1\times n_2$ matrices	29
$\mathbb{R}^{n_1\times n_2}\left[\xi,\xi^{-1}\right]$	the set of real Laurent-polynomial $n_1\times n_2$ matrices	20
$\mathbb{R}^{\bullet\times n}\left[\xi,\xi^{-1}\right]$	the set of real Laurent-polynomial matrices with n columns	19
(A,B,C,D)	time-invariant state space system	3
$\left(A^T,C^T,B^T,D^T\right)$	dual of (A,B,C,D)	13
$(A\left(\cdot\right),B\left(\cdot\right),C\left(\cdot\right),D\left(\cdot\right))$	periodic state space system	3
Σ_s	periodic state space system	3
$\Sigma_t\equiv(A_t,B_t,C_t,D_t)$	the P time-invariant state space systems associated with Σ_s	5
$\phi_A\left(k,k_0\right)$	the state transition matrix	6
KR	kernel representation	19
LVR	latent variable representation	21
IR	image representation	21
P-PKR	P-periodic kernel representation	47
P-PLVR	P-periodic latent variable representation	56
P-PIR	P-periodic image representation	59

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Index